MA 16200 Quiz 20 Solutions

Approximating Functions with Polynomials, Properties of Power Series

1. Find the interval of convergence for the following series:

$$\sum_{n=1}^{\infty}(-1)^n \frac{(x-2)^n}{n}$$

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-2)^{n+1}}{n+1}\cdot\frac{n}{(-1)^n(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{n(x-2)}{n+1}\right| = |x-2| < 1$$

So $-1 < x-2 < 1 \Rightarrow 1 < x < 3$.
Check endpoints:
$x = 1$:

$$\sum_{n=1}^{\infty}(-1)^n \frac{(-1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n} = \sum_{n=1}^{\infty}\frac{1}{n} \text{ diverges by p-test}$$

$x = 3$:

$$\sum_{n=1}^{\infty}(-1)^n \frac{(1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} \text{ converges by AST}$$

Interval: $(1, 3]$

2. Find the radius of convergence for the following series:

$$\sum_{k=1}^{\infty}\frac{k^2(x+1)^k}{(k+2)3^{2k}}$$

Ration Test:

$$\lim_{k\to\infty}\left|\frac{(k+1)^2(x+1)^{k+1}}{(k+3)3^{2k+2}}\cdot\frac{(k+2)3^{2k}}{k^2(x+1)^k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)^2(k+2)(x+1)}{k^2(k+3)3^2}\right| = \frac{|x+1|}{9} < 1$$

So $|x+1| < 9 \Rightarrow -9 < x+1 < 9 \Rightarrow -10 < x < 8$.
You should identify from the above power series that the center is $a = -1$. Therefore, the radius is $R = 9$.

3. Find the remainder term R_n in the nth-order Taylor polynomial centered at $a = 0$ for the given function. Express the result for a general value of n.

$$f(x) = e^{2x}$$

Now, use the remainder term to estimate the absolute error $|f(-0.1) - p_2(-0.1)|$.
Recall:

$$R_n(x) = |f(x) - p_n(x)| = \frac{f^{(n+1)}(c)|x-a|^{n+1}}{(n+1)!} \qquad c \text{ between } a \text{ and } x$$

and

$$R_n(x) \leq \frac{M|x-a|^{n+1}}{(n+1)!}$$

where $|f^{(n+1)}(c)| \leq M$.
$f'(x) = 2e^{2x}$, $f''(x) = 4e^{2x}$, $f'''(x) = 8e^{2x}$, $f^{(n+1)}(x) = 2^{n+1}e^{2x}$
Thus:

$$R_n(x) = \frac{2^{n+1}e^{2c}|x|^{n+1}}{(n+1)!} \qquad -0.1 \leq c \leq 0$$

Next, it is given that $n = 2$ and $x = -0.1$:

$$|f(-0.1) - p_2(-0.1)| = R_2(-0.1) \leq \frac{M|-0.1|^3}{3!}$$

where

$$|f^{(3)}(c)| = |8e^{2c}| \leq M$$

$f^{(3)}(c)$ is maximized at $c = 0$. Therefore $M = 8$. Thus

$$R_2(-0.1) \leq \frac{8|-0.1|^3}{3!} = \frac{4(0.1)^3}{3}$$