Chaos and Fractals

by John Como
Benoit B. Mandelbrot

- Coined the term fractal from the Latin word *fractus* meaning broken and irregular.
- Created the now famous Mandelbrot set in the late 1950’s with IBM.
What is a fractal?

• A fractal has a very fine structure.
• Too irregular to be described by regular calculus or traditional geometry.
• Has self-similarity.
• Can have a non-integer dimension.
• Can be described by a simple iterative formula.
• Can often have a natural appearance.
Simple Fractals
M.C. Escher

• Circle Limits IV can be considered a fractal due to its self-similarity.
• If you zoom in on the edge of the circle, the same picture is repeated to infinity.
Durer’s Pentagons

- Start with a pentagon and divide it into six like pentagons (first iteration).
- Divide those into six pentagons etc.
Koch Curve

• The Koch snowflake is a fractal whose area is enclosed by an infinite perimeter!
• The Kock curve area increases per iteration but converges at a finite number.

\[
A_n = A_0 + A_0 \cdot \frac{3}{9} + A_0 \cdot \frac{27}{81} + A_0 + \frac{48}{729} + \cdots
\]

\[
= A_0 \left(1 + \frac{1}{3} \left(\frac{4}{9}\right)^0 + \frac{1}{3} \left(\frac{4}{9}\right)^1 + \frac{1}{3} \left(\frac{4}{9}\right)^2 + \cdots + \frac{1}{3} \left(\frac{4}{9}\right)^n\right)
\]

\[
= A_0 \left(1 + \frac{1}{3} \sum_{k=0}^{n} \left(\frac{4}{9}\right)^k\right)
\]

\[
A \equiv \lim_{n \to \infty} A_n = A_0 \left(1 + \frac{1}{3} \sum_{k=0}^{\infty} \left(\frac{4}{9}\right)^k\right)
\]

\[
= A_0 \left(1 + \frac{1}{3} \frac{1}{1 - \frac{4}{9}}\right) = \frac{8}{5} A_0
\]

\[
A = \frac{8}{5} A_0
\]
• The perimeter is infinite!
  – N is the number of sides
  – L is the length of one segment
  – P is the perimeter

\[
N_0 = 3 \\
N_1 = 12 = 4 \cdot 3 \\
N_2 = 48 = 4^2 \cdot 3 \\
\vdots \\
N_k = 4^k \cdot 3
\]

\[
P_k = N_k L_k = 4^k \cdot 3 \cdot \left(\frac{1}{3}\right)^k
\]

\[
L_k = \left(\frac{1}{3}\right)^k \\
= 3 \cdot \left(\frac{4}{3}\right)^k
\]
How to make a fractal...

• Start with a recursion formula.

• Example: The different Julia sets are given by

\[ z_{n+1} = z_n^2 + c \]
How to make a fractal...

- Iterate different $z$ values for the entire complex plain.
- If, after a set number of iterations, the iterates do not pass a critical magnitude, the point is colored black.
- If the iterates do pass the critical magnitude, the point is colored corresponding to the number of iterations it went through.
Julia sets for different $c$ values

$c = -1 + i0$

$c = 0.3 + i0.6$
The Mandelbrot Set

• Acts as a ‘dictionary’ for the Julia sets.
The Mandelbrot Set

- Same iteration formula, however it contains all the Julia sets within it by connecting them.
Undoubtedly the most famous fractal is..
What are fractals used for?

• Fractal pictures bridge the gap between our real world and the mathematical world.
• Are used in Hollywood
  – Ex. Were used in *Star Trek II: The Wrath of Khan* for the genesis planet.
• Are used in video graphic design to create landscapes.
• Make beautiful art.
• (M’art’hematicians)
Application

- Fractals describe this coastline.
Application

• These fractals look like real trees.
• Fractal images of leaves and mountains.
From chaos comes order and beauty.
Thanks to…

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