Solutions to Exercises (Sections 1.11 - 1.12)

Section 1.11
Exercise 1.11.1
(a)
\[ p \rightarrow q \]
\[ q \rightarrow r \]
\[ \neg r \]
\[ \therefore \neg p \]
Solution
1. \( q \rightarrow r \) Hypothesis
2. \( p \rightarrow q \) Hypothesis
3. \( p \rightarrow r \) Hypothetical syllogism, 1, 2
4. \( \neg r \) Hypothesis
5. \( \neg p \) Modus tollens, 3, 4.

(b)
\[ p \rightarrow (q \land r) \]
\[ \neg q \]
\[ \therefore \neg p \]
Solution
1. \( \neg q \) Hypothesis
2. \( \neg q \lor \neg r \) Addition, 1
3. \( \neg(q \land r) \) De Morgan’s law, 2
4. \( p \rightarrow (q \land r) \) Hypothesis
5. \( \neg p \) Modus tollens, 3, 4

(c)
\[ (p \land q) \rightarrow r \]
\[ \neg r \]
\[ q \]
\[ \therefore \neg p \]
Solution
1. \( (p \land q) \rightarrow r \) Hypothesis
2. \( \neg r \) Hypothesis
3. \( \neg(p \land q) \) Modus tollens, 1, 2.
4. \( \neg p \lor \neg q \) De Morgan’s law, 3
5. \( q \) Hypothesis
6. \( \neg\neg q \)  Double negation law, 5.
7. \( \neg p \)  Disjunctive syllogism, 4, 6.

(d)
\[(p \lor q) \rightarrow r\]
p
\(\therefore r\)
Solution
1. \((p \lor q) \rightarrow r\)  Hypothesis
2. \(p\)  Hypothesis
3. \(p \lor q\)  Addition, 2.
4. \(r\)  Modus ponens, 1, 3.

(e)
\(p \lor q\)
\(\neg p \lor r\)
\(\neg q\)
\(\therefore r\)
Solution
1. \(p \lor q\)  Hypothesis
2. \(\neg p \lor r\)  Hypothesis
3. \(q \lor r\)  Resolution, 1, 2.
4. \(\neg q\)  Hypothesis
5. \(r\)  Disjunctive syllogism, 3, 4.

(f)
\(p \rightarrow q\)
\(r \rightarrow u\)
\(p \land r\)
\(\therefore q \land u\)
Solution
1. \(p \rightarrow q\)  Hypothesis
2. \(p \land r\)  Hypothesis
3. \(p\)  Simplification, 2
4. \(q\)  Modus ponens, 1, 3.
5. \(r \rightarrow u\)  Hypothesis
6. \(r\)  Simplification, 2
7. \(u\)  Modus ponens, 5, 6
8. \(q \land u\)  Conjunction, 4, 7.
Exercise 1.11.2
Some of the rules of inference can be proven using the other rules of inference and the laws of propositional logic.

(a)
One of the rules of inference is Modus tollens:

\[ p \rightarrow q \]
\[ \neg q \]
\[ \therefore \neg p \]

Prove that Modus tollens is valid using the laws of propositional logic and any of the other rules of inference besides Modus tollens. (Hint: you will need one of the conditional identities from the laws of propositional logic).

Solution
1. \( p \rightarrow q \) Hypothesis
2. \( \neg p \lor q \) Conditional identity, 1
3. \( \neg q \) Hypothesis
4. \( \neg p \) Disjunctive syllogism, 2, 3

(b)
One of the rules of inference is Modus ponens:

\[ p \rightarrow q \]
\[ p \]
\[ \therefore q \]

Prove that Modus ponens is valid using the laws of propositional logic and any of the other rules of inference besides Modus ponens. (Hint: you will need one of the conditional identities from the laws of propositional logic).

Solution
1. \( p \rightarrow q \) Hypothesis
2. \( \neg p \lor q \) Conditional identity, 1
3. \( p \) Hypothesis
4. \( \neg \neg p \) Double negation, 3
5. \( q \) Disjunctive syllogism, 2, 4
(c) One of the rules of inference is Disjunctive syllogism:

\[ p \lor q \]
\[ \neg p \]
\[ \therefore q \]

Prove that Disjunctive syllogism is valid using the laws of propositional logic and any of the other rules of inference besides Disjunctive syllogism. (Hint: you will need one of the conditional identities from the laws of propositional logic).

Solution
1. \( p \lor q \) Hypothesis
2. \( \neg \neg p \lor q \) Double negation, 1
3. \( \neg p \rightarrow q \) Conditional identity, 2
4. \( \neg p \) Hypothesis
5. \( q \) Modus ponens, 3, 4

(d) One of the rules of inference is Resolution:

\[ p \lor q \]
\[ \neg p \lor r \]
\[ \therefore q \lor r \]

Prove that Resolution is valid using the laws of propositional logic and any of the other rules of inference besides Resolution. (Hint: you will need one of the conditional identities from the laws of propositional logic).

Solution
1. \( p \lor q \) Hypothesis
2. \( q \lor p \) Commutative law, 1
3. \( \neg \neg q \lor p \) Double negation, 2
4. \( \neg q \rightarrow p \) Conditional identity, 3
5. \( \neg p \lor r \) Hypothesis
6. \( p \rightarrow r \) Conditional identity, 5
7. \( \neg q \rightarrow r \) Hypothetical syllogism, 4, 6
8. \( \neg \neg q \lor r \) Conditional identity, 7
9. \( q \lor r \) Double negation, 8
Exercise 1.11.3
Prove that each argument is valid by replacing each proposition with a variable to obtain the form of the argument. Then use the rules of inference to prove that the form is valid.

(a)
If I drive on the freeway, I will see the fire.
I will drive on the freeway or take surface streets (or both).
I am not going to take surface streets.
∴ I will see the fire.
Solution
w: I drive on the freeway
f: I will see the fire
s: I drive on surface streets
The form of the argument is

\[ w \rightarrow f \]
\[ w \lor s \]
\[ \neg s \]
∴ \[ f \]

1. \[ w \lor s \] Hypothesis
2. \[ \neg s \] Hypothesis
3. \[ w \] Disjunctive syllogism, 1, 2
4. \[ w \rightarrow f \] Hypothesis
5. \[ f \] Modus ponens, 3, 4

(b)
If it was not foggy or it didn't rain (or both), then the race was held and there was a trophy ceremony.
The trophy ceremony was not held.
∴ It rained.
Solution
f: it was foggy
r: it rained
h: the race was held
t: the trophy ceremony was held
The form of the argument is

\[ (\neg f \lor \neg r) \rightarrow (h \land t) \]
\[ \neg t \]
∴ r
1. \( \neg t \) Hypothesis
2. \( \neg t \lor \neg h \) Addition, 1
3. \( \neg(t \land h) \) De Morgan's law, 2
4. \( \neg(h \land t) \) Commutative law, 3
5. \( (\neg f \lor \neg r) \rightarrow (h \land t) \) Hypothesis
6. \( \neg(\neg f \lor \neg r) \) Modus tollens, 4, 5
7. \( \neg f \land \neg r \) De Morgan's law, 6
8. \( \neg \neg r \) Simplification, 7
9. \( r \) Double negation, 8

(c)
If I work out hard, then I am sore.
If I am sore, I take an aspirin.
I did not take an aspirin.
∴ I did not work out hard.

Solution
h: I work out hard
s: I am sore
a: I took an aspirin
The form of the argument is

\[ h \rightarrow s \]
\[ s \rightarrow a \]
\[ \neg a \]
∴ \( \neg h \)
1. \( h \rightarrow s \) Hypothesis
2. \( s \rightarrow a \) Hypothesis
3. \( h \rightarrow a \) Hypothetical syllogism, 1, 2
4. \( \neg a \) Hypothesis
5. \( \neg h \) Modus tollens, 3, 4.

Section 1.12
Exercise 1.12.1
Prove that the given argument is valid. First find the form of the argument by defining predicates and expressing the hypotheses and the conclusion using the predicates. Then use the rules of inference to prove that the form is valid.

(a)
The domain of discourse is the set of musicians in an orchestra.

Everyone practices hard or plays badly (or both).
Someone does not practice hard.
\[ \therefore \text{Someone plays badly.} \]

Solution

\[ P(x) : x \text{ practices hard} \]
\[ B(x) : x \text{ plays badly} \]

The form of the argument is

\[ \forall x (P(x) \lor B(x)) \]
\[ \exists x \neg P(x) \]
\[ \therefore \exists x B(x) \]

1. \[ \exists x \neg P(x) \] Hypothesis
2. \[ (c \text{ is a specific musician in the orchestra}) \land \neg P(c) \] Existential instantiation, 1
3. \[ \neg P(c) \] Simplification, 2
4. \[ c \text{ is a specific musician in the orchestra} \] Simplification, 2
5. \[ \forall x (P(x) \lor B(x)) \] Hypothesis
6. \[ P(c) \lor B(c) \] Universal instantiation, 4, 5
7. \[ B(c) \] Disjunctive syllogism, 3, 6
8. \[ \exists x B(x) \] Existential generalization, 4, 7

(b)
The domain of discourse is the set of people who live in a city. Linda lives in the city.

Linda lives in the city.
Linda owns a Ferrari.
Everyone who owns a Ferrari has gotten a speeding ticket.
\[ \therefore \text{Linda has gotten a speeding ticket.} \]

Solution

\[ F(x) : x \text{ owns a Ferrari} \]
\[ S(x) : x \text{ has gotten a speeding ticket} \]

The form of the argument is

Linda lives in the city
\[ F(Linda) \]
\[ \forall x (F(x) \rightarrow S(x)) \]
\[ \therefore S(Linda) \]
1. Linda lives in the city Hypothesis
2. \( \forall x (F(x) \rightarrow S(x)) \) Hypothesis
3. \( F(Linda) \rightarrow S(Linda) \) Universal instantiation, 1, 2
4. \( F(Linda) \) Hypothesis
5. \( S(Linda) \) Modus ponens, 3, 4

(c)
The domain of discourse is the set of all paintings.

All of the paintings by Matisse are beautiful.
The museum has a painting by Matisse.
\[ \therefore \text{ The museum has a beautiful painting.} \]

Solution
B(x): x is beautiful
M(x): x is by Matisse
S(x): The museum has x
The form of the argument is

\[
\forall x \ (M(x) \rightarrow B(x)) \\
\exists x \ (S(x) \land M(x)) \\
\therefore \exists x \ (B(x) \land S(x))
\]

1. \( \exists x \ (S(x) \land M(x)) \) Hypothesis
2. c is a particular painting Element definition
3. \( S(c) \land M(c) \) Existential instantiation, 1, 2
4. \( S(c) \) Simplification, 3
5. \( M(c) \) Simplification, 3
6. \( \forall x \ (M(x) \rightarrow B(x)) \) Hypothesis
7. \( M(c) \rightarrow B(c) \) Universal instantiation, 2, 6
8. \( B(c) \) Modus ponens, 5, 7
9. \( B(c) \land S(c) \) Conjunction, 4, 8
10. \( \exists x \ (B(x) \land S(x)) \) Existential generalization, 2, 9

(d)
The domain is the set of students at an elementary school.

Every student who has a permission slip can go on the field trip.
Every student has a permission slip.
\[ \therefore \text{ Every student can go on the field trip.} \]

Solution
P(x): x has a permission slip
F(x): x can go on the field trip
The form of the argument is:

\[ \forall x \ (P(x) \rightarrow F(x)) \]
\[ \forall x \ P(x) \]
\[ \therefore \ \forall x \ F(x) \]

1. c is an arbitrary student at the school \hspace{1cm} \text{Element definition}
2. \[ \forall x \ (P(x) \rightarrow F(x)) \] \hspace{1cm} \text{Hypothesis}
3. P(c) \rightarrow F(c) \hspace{1cm} \text{Universal instantiation, 1, 2}
4. \[ \forall x \ P(x) \] \hspace{1cm} \text{Hypothesis}
5. P(c) \hspace{1cm} \text{Universal instantiation, 1, 4}
6. F(c) \hspace{1cm} \text{Modus ponens, 3, 5}
7. \[ \forall x \ F(x) \] \hspace{1cm} \text{Universal generalization, 1, 6}

\( (e) \)
The domain of discourse is the set of students at a university.

Larry is a student at the university.
Hubert is a student at the university.
Larry and Hubert are taking Boolean Logic.
Any student who takes Boolean Logic can take Algorithms.
\[ \therefore \] Larry and Hubert can take Algorithms.

Solution

B(x): x is taking Boolean Logic
A(x): x can take Algorithms
The form of the argument is

Larry is a student at the university.
Hubert is a student at the university.
B(Larry) \land B(Hubert)
\[ \forall x \ (B(x) \rightarrow A(x)) \]
\[ \therefore \ A(Larry) \land A(Hubert) \]

1. Larry is a student at the university. \hspace{1cm} \text{Hypothesis}
2. Hubert is a student at the university. \hspace{1cm} \text{Hypothesis}
3. B(Larry) \land B(Hubert) \hspace{1cm} \text{Hypothesis}
4. B(Larry) \hspace{1cm} \text{Simplification, 3}
5. \[ \forall x \ (B(x) \rightarrow A(x)) \] \hspace{1cm} \text{Hypothesis}
6. B(Larry) \rightarrow A(Larry) \hspace{1cm} \text{Universal instantiation, 1, 5}
7. A(Larry) \hspace{1cm} \text{Modus ponens, 4, 6}
Exercise 1.12.2
Which of the following arguments are valid? Explain your reasoning.

(a) I have a student in my class who is getting an A. Therefore, John, a student in my class is getting an A.

   Solution: Not valid. Consider the scenario in which there is a student in the class who is not John and is getting an A. Also suppose that John is not getting an A. Under this scenario, the hypothesis is true and the conclusion is false. Therefore, the argument is not valid.

(b) Every girl scout who sells at least 50 boxes of cookies will get a prize. Suzy, a girl scout, got a prize. Therefore Suzy sold 50 boxes of cookies.

   Solution: Not valid. The first hypothesis is that $\forall x (S(x) \rightarrow P(x))$, where the predicate $S(x)$ means that $x$ has sold at least 50 boxes of cookies and $P(x)$ means that $x$ got a prize. Consider a scenario in which there is a girl scout troop for which $\forall x (S(x) \rightarrow P(x))$ is true. Furthermore, there is a girl named Suzy in the troop such that $P(Suzy)$ is true and $S(Suzy)$ is false. Then all the hypotheses are true but the conclusion is false. Therefore the argument is not valid.

Exercise 1.12.3
Show that the given argument is invalid by giving values for the predicates $P$ and $Q$ over the domain $\{a, b\}$.

(a)
$\forall x (P(x) \rightarrow Q(x))$
$\exists x \neg P(x)$
$\therefore \exists x \neg Q(x)$

Solution
∀ x (P(x) → Q(x)) is true because P(x) is false for both inputs a and b. ∃ x ¬P(x) is true since both a and b are examples. However, since Q(a) = Q(b) = T, ∃ x ¬Q(x) is false. Therefore both hypotheses are true and the conclusion is false.

(b)
∃ x (P(x) ∨ Q(x))
∃ x ¬Q(x)
∴ ∃ x P(x)

Solution

P Q
a F T
b F T

∃ x (P(x) ∨ Q(x)) is true because P(a) ∨ Q(a) is true. ∃ x ¬Q(x) is true since ¬Q(b) is true. However, since P(a) = P(b) = F, ∃ x P(x) is false. Therefore both hypotheses are true and the conclusion is false.

Exercise 1.12.4
Determine whether each argument is valid. If the argument is valid, give a proof using the laws of logic. If the argument is invalid, give values for the predicates P and Q over the domain {a, b} that demonstrate the argument is invalid.

(a)
∃ x (P(x) ∧ Q(x))
∴ ∃ x Q(x) ∧ ∃ x P(x)

Solution
Valid.

1. ∃ x (P(x) ∧ Q(x)) Hypothesis
2. (c is a particular element) ∧ (P(c) ∧ Q(c)) Existential instantiation, 1
3. P(c) ∧ Q(c) Simplification, 2
4. P(c) Simplification, 3
5. c is a particular element Simplification, 2
6. ∃ x P(x) Existential generalization, 4, 5
7. Q(c) Simplification, 3
8. ∃ x Q(x) Existential generalization, 5, 7
9. \( \exists x \, P(x) \land \exists x \, Q(x) \) Conjunction, 6, 8

(b) 
\( \exists x \, Q(x) \land \exists x \, P(x) \)
\[ \therefore \exists x \, (P(x) \land Q(x)) \]

Solution
Not valid.

\[
\begin{array}{cc}
P & Q \\
a & F & T \\
b & T & F \\
\end{array}
\]

\( \exists x \, Q(x) \) is true because \( Q(a) \) is true. \( \exists x \, P(x) \) is true because \( P(b) \) is true. However, neither \( P(a) \land Q(a) \) nor \( P(b) \land Q(b) \) is true, so \( \exists x \, (P(x) \land Q(x)) \) is false. Therefore the hypothesis is true and the conclusion is false.

(c) 
\( \forall x \, (P(x) \land Q(x)) \)
\[ \therefore \forall x \, Q(x) \land \forall x \, P(x) \]

Solution
Valid.

1. \( \forall x \, (P(x) \land Q(x)) \) Hypothesis
2. \( c \) is an arbitrary element Element definition
3. \( P(c) \land Q(c) \) Universal instantiation, 1, 2
4. \( P(c) \) Simplification, 3
5. \( \forall x \, P(x) \) Universal generalization, 2, 4
6. \( Q(c) \) Simplification, 3
7. \( \forall x \, Q(x) \) Universal generalization, 2, 6
8. \( \forall x \, P(x) \land \forall x \, Q(x) \) Conjunction, 5, 7

(d) 
\( \forall x \, (P(x) \lor Q(x)) \)
\[ \therefore \forall x \, Q(x) \lor \forall x \, P(x) \]

Solution
Not valid.

\[
\begin{array}{cc}
P & Q \\
a & F & T \\
b & T & F \\
\end{array}
\]
P(a) ∨ Q(a) and P(b) ∨ Q(b) are both true, so ∀x (P(x) ∨ Q(x)) is true. However, neither ∀x P(x) nor ∀x Q(x) is true. Therefore the hypothesis is true and the conclusion is false.