12.2.2 Example: Spacing of Blood Vessels in Artificial Tissue

Development of artificial tissues requires designing the vascular network. Consider capillaries as long cylindrical vessels of diameter 10 µm and oxygen transport to be one-dimensional, in the radial direction, as shown in Figure 12.16. The spacing between capillaries needs to be small enough to make sure sufficient oxygen is available to all tissue areas. Due to symmetry with the neighboring capillaries, the concentration profile can be considered symmetric at a particular radius, 200 µm, resulting in a zero flux condition. On the capillary wall, a constant oxygen concentration of 130 µM is maintained by the flowing blood. The rate of oxygen consumption in the tissue is 30 µM/s. The diffusivity of oxygen in the tissue is $1.7 \times 10^{-9}$ m$^2$/s. 1) Show the governing equation for steady-state oxygen transport with only the needed terms. 2) Show all the boundary conditions needed to solve the equation in step 1. 3) Solve the governing equation to obtain the concentration profile. 4) Assume a minimum concentration of 65 µM is needed in the tissue. Write the equation (with all appropriate values inserted, no need to solve) from which you can calculate the spacing between capillaries that ensures this minimum oxygen concentration in the tissue.

Solution

Understanding and formulating the problem 1) What is the process? Oxygen is transported from the blood into the tissue with its simultaneous consumption into the tissue. 2) What are we solving for? Concentration of oxygen as a function of position in the tissue. 3) Schematic and given data: A schematic is shown in Figure 12.17 with some of the given data superimposed on it. 4) Assumptions: Since the tube wall is thin, it can be approximated as a slab, following the discussion surrounding Eq. 4.48 on page 88.

Generating and selecting among alternate solutions 1) What solutions are possible? For steady-state mass transfer, the solutions we have available are shown in the solution chart in Figure 12.19. 2) What approach is likely to work? Using this chart, since there is a reaction (consumption of oxygen), we either derive from scratch using the governing equation and boundary conditions or choose a solution already derived in the text if conditions (governing equation, boundary conditions, and order of reaction) match. Since the derivation in Section 12.2 on page 424 is for a slab geometry and first-order reaction, it does not match the problem being solved here (which is a cylindrical geometry and has zeroth-order reaction). Thus, we have to derive the solution from scratch.

Implementing the chosen solution Since the blood vessels are cylindrical, we have
to start from the governing diffusion equation in cylindrical coordinates,

\[
\frac{\partial \phi}{\partial t} = D_{AB} \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial c}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \right) + r_A
\]

where the respective terms have been dropped out based on the reasons provided in the problem description. The final governing equation is

\[
D_{AB} \frac{d}{dr} \left( r \frac{dc}{dr} \right) + r_A = 0 \quad (12.26)
\]

This is a second-order equation that needs boundary conditions to solve. These are

\[
c = 130 \mu m \text{ at } r = R_i \quad (12.27)
\]

\[
\frac{dc}{dr} = 0 \text{ at } r = R_o \quad (12.28)
\]

Integrating Eq. 12.26,

\[
\frac{d}{dr} \left( r \frac{dc}{dr} \right) = -r \frac{r_A}{D_{AB}}
\]

\[
\int d \left( r \frac{dc}{dr} \right) = -r_A \frac{r^2}{D_{AB}} \int r \, dr
\]

\[
r \frac{dc}{dr} = -\frac{r_A R_i^2}{2 D_{AB}} + k_1
\]

Using the second boundary condition above (Eq. 12.28), we get

\[
0 = -\frac{r_A R_o^2}{D_{AB}} + k_1
\]

\[
k_1 = \frac{r_A R_o^2}{D_{AB}}
\]
Plugging in \( k_1 \) and integrating again,

\[
\begin{align*}
\frac{r}{dc}dr &= -\frac{r_A}{D_{AB}} \frac{r^2}{2} + \frac{r_A}{D_{AB}} \frac{R_o^2}{2} \\
\frac{dc}{dr} &= \frac{r_A}{D_{AB}} \frac{r}{2} + \frac{r_A}{D_{AB}} \frac{R_o^2}{2} \frac{1}{r} \\
\int dc &= -\frac{r_A}{D_{AB}} \int \frac{r}{2} dr + \frac{r_A}{D_{AB}} \frac{R_o^2}{2} \int \frac{1}{r} dr \\
c &= -\frac{r_A}{D_{AB}} \frac{r^2}{4} + \frac{r_A}{D_{AB}} \frac{R_o^2}{2} \ln r + k_2 \quad (12.29)
\end{align*}
\]

Using the first boundary condition (at \( r = R_i \)) above (Eq. 12.27),

\[
130 = -\frac{r_A}{D_{AB}} \frac{R_i^2}{4} + \frac{r_A}{D_{AB}} \frac{R_o^2}{2} \ln R_i + k_2
\]

which can be subtracted from 12.29 to eliminate \( k_2 \) and thus obtain the expression for concentration, \( c \),

\[
\begin{align*}
c - 130 &= -\frac{r_A}{D_{AB}} \frac{r^2 - R_i^2}{4} + \frac{r_A}{D_{AB}} \frac{R_o^2}{2} \ln \frac{r}{R_i} \\
c &= 130 - \frac{r_A}{D_{AB}} \frac{r^2 - R_i^2}{4} + \frac{r_A}{D_{AB}} \frac{R_o^2}{2} \ln \frac{r}{R_i} \quad (12.30)
\end{align*}
\]

2) Let \( r' \) be the distance at which we have the minimum oxygen. Thus we can write (note \( r_A \) is negative since oxygen is being consumed)

\[
65 = 130 + \frac{30 \left( r'^2 - (5 \times 10^{-6})^2 \right)}{4 \times 1.7 \times 10^{-9}} - \frac{30 \times (200 \times 10^{-6})^2}{2 \times 1.7 \times 10^{-9}} \ln \left( \frac{r'}{5 \times 10^{-6}} \right)
\]

\[
65 = 130 + 4.41 \times 10^9 \left( r'^2 - 25 \times 10^{-12} \right) - 352.94 \ln \left( r'/(5 \times 10^{-6}) \right)
\]

from which \( r' \) is calculated as \( r' = 6.01 \times 10^{-6} \text{m} = 6.01 \mu\text{m} \).

**Evaluating and interpreting the solution**

1) *Does the concentration profile make sense?*

The solution given by Eq. 12.30 is shown in Figure 12.18. It starts from the maximum value 130 \( \mu\text{M} \) at the edge of the blood vessel and approaches zero due to the constant rate of consumption (zeroth order), so this general trend makes sense. The solution is assumed to be not valid for negative values of \( c \).

2) *What do we learn?*

The region over which sufficient oxygen can diffuse when it is also being consumed can be rather small.