The Finite Element Method
– Part I –

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Introduction to Numerical Methods for ENG & CS
(Mathematics IV)

Spring Term 2016
Introduction & Today’s Scope
FDs are nice, but often we have another discretization structure of the domain. For instance, CAD prototyping gives us a triangulation of a new part that looks very different from the finite difference (FD) discretization we discussed:

What do we do now?
Today, we will focus on the weak formulation of a classical PDE.

Today's topics:

- Weak formulation and multiplication by test functions
- Required mathematical tools and function spaces
- Prerequisites for the discretization: affine functions on a triangle and the concept of a triangulation

Next time, we discuss:

- Discretization/ triangulation of a domain
- Discretization of the function spaces involved in the weak formulation

...to finally, the complete picture of the finite element method.
From the Strong Form to the Weak Form
Like in for the finite differences method the first thing we have to describe is the geometry (the physical setting of the problem): The domain $\Omega$ and the Dirichlet $\Gamma_D$ and Neumann boundaries $\Gamma_N$, with $\Gamma_D \cup \Gamma_N = \Gamma = \partial \Omega$.

You can think in more mechanical terms as follows: the Dirichlet boundary is where displacements are given as data; the Neumann boundary is where normal stresses are given as data.
In the domain we will have an elliptic partial differential equation of second order (to keep notation as simple as possible) and on the boundary we will impose conditions on the solution:

\[
\begin{cases}
-\Delta u(x) + c \cdot u(x) = f(x), & \text{in } \Omega, \\
u(x) = g_0(x), & \text{in } \Gamma_D, \\
\partial_n u(x) = g_1(x), & \text{in } \Gamma_N.
\end{cases}
\]

There are new many things here, so let’s go step by step:

1. The unknown is a (scalar valued) twice-differentiable function \( u \) defined on the domain \( \Omega \).

2. \( c \) is a non-negative constant value. It is put there to make clear two different terms when we go on to see the numerical approximation.

This equation is usually called a reaction-diffusion equation. The diffusion term is given by \(-\Delta u\) and the reaction term, when \( c > 0 \), is \( cu \).
In the domain we will have an elliptic partial differential equation of second order (to keep notation as simple as possible) and on the boundary we will impose conditions on the solution:

\[
\begin{cases}
    -\Delta u(x) + c \cdot u(x) &= f(x), & \text{in } \Omega, \\
    u(x) &= g_0(x), & \text{in } \Gamma_D, \\
    \partial_n u(x) &= g_1(x), & \text{in } \Gamma_N.
\end{cases}
\]

There are new many things here, so let’s go step by step:

3. \(f\) is a given continuous function on \(\Omega\). It corresponds to source terms in the equation. It can be considered as a surface density of forces.

4. There are two functions \(g_0\) and \(g_1\) given on the two different parts of the boundary. They will play very different roles in our formulation. As a general rule, we will demand that \(g_0\) is a continuous function, whereas \(g_1\) will be allowed to be discontinuous.
The problem, written in strong form
(3/3)

In the domain we will have an elliptic partial differential equation of second order (to keep notation as simple as possible) and on the boundary we will impose conditions on the solution:

$$\begin{cases} 
-\Delta u(x) + c \cdot u(x) &= f(x), \quad \text{in } \Omega, \\
u(x) &= g_0(x), \quad \text{in } \Gamma_D, \\
\partial_n u(x) &= g_1(x), \quad \text{in } \Gamma_N.
\end{cases}$$

There are new many things here, so let’s go step by step:

5. The symbol $\partial_n$ denotes the exterior normal derivative, that is,

$$\partial_n u(x) = \langle \nabla u(x), n(x) \rangle,$$

where $n$ is the unit normal vector on points of $\Gamma$ pointing always outwards and $\nabla u$ is the gradient of $u$. 
The approach to solve this problem above with the **Finite Element Method** is based upon writing it in a completely different form, which is sometimes called **weak** or **variational form**.

At the beginning it can look confusing to see all this if you are not used to advanced mathematics in continuum mechanics or physics. We are just going to show here how the formulation is obtained and what it looks like at the end.

The most important theorem in this process or reformulating the problem is **Green’s Theorem**:

\[
\int_{\Omega} (\Delta u) v \, dx + \int_{\Omega} \langle \nabla u, \nabla v \rangle \, dx = \int_{\Gamma_N} (\partial_n u) v \, ds.
\]
\[ \int_{\Omega} (\Delta u) v \, dx + \int_{\Omega} \langle \nabla u, \nabla v \rangle \, dx = \int_{\Gamma} (\partial_n u) v \, ds. \]

Note that there are two types of integrals in this formula. Both integrals in the left-hand side are domain integrals in \( \Omega \), whereas the integral in the right-hand side is a line integral on the boundary \( \Gamma \).

By the way, the result is also true in three dimensions. In that case, domain integrals are volume integrals and boundary integrals are surface integrals. This leads to the set-up of the **Finite Volume Method**.
Green’s Theorem is in fact a simple consequence of the Divergence Theorem:

\[ \int_{\Omega} (\text{div}(p(x))) \, u \, dx + \int_{\Omega} \langle p(x), \nabla v \rangle \, dx = \int_{\Gamma} \langle p(x), n \rangle \, v \, ds. \]

Here, \( \text{div}(p) \) is the divergence of the vector field \( p \), that is, if \( p = (p_1, p_2) \)

\[ \text{div}(p) = \frac{\partial p_1(x)}{\partial x_1} + \frac{\partial p_2(x)}{\partial x_2}. \]

If we take \( p = \nabla u \) we obtain Green’s Theorem.

**Remark:** In 1-dimensions it boils down to partial integration and the fundamental theorem of calculus.
Starting point for the set-up of the weak formulation is the strong form of the PDE multiplied with a test function \( v \) (wait for comments on \( v \) as the section progresses):

\[
-\Delta u(x) + c \cdot u(x) = f(x)
\]

\[
\downarrow
\]

\[
-\Delta u(x) \cdot v(x) + c \cdot u(x) \cdot v(x) = f(x) \cdot v(x).
\]

The next step is an averaging over the domain, i.e. taking the integral over \( \Omega \):

\[
-\int_{\Omega} \Delta u \cdot v \, dx + \int_{\Omega} c \cdot u \cdot v \, dx = \int_{\Omega} f \cdot v \, dx.
\]

Next, we use Green’s Theorem to substitute \(-\int_{\Omega} \Delta u(x) v \, dx\):

\[
-\int_{\Omega} (\Delta u) v \, dx \overset{\text{Green}}{=} \int_{\Omega} \langle \nabla u , \nabla v \rangle \, dx - \int_{\Gamma} (\partial_n u) v \, ds.
\]
Thus, we have:

\[
\int_{\Omega} \langle \nabla u, \nabla v \rangle \, dx - \int_{\Gamma} (\partial_n u) v \, ds + \int_{\Omega} c \cdot u \cdot v \, dx = \int_{\Omega} f \cdot v \, dx,
\]

or, with \( v(x) = 0 \) on \( \Gamma_D \)

\[
\int_{\Omega} \langle \nabla u, \nabla v \rangle \, dx + c \cdot \int_{\Omega} u \cdot v \, dx = \int_{\Omega} f \cdot v \, dx + \int_{\Gamma_N} \left( \partial_n u \right) v \, ds
= g_1
\]

\[
= \int_{\Omega} f \cdot v \, dx + \int_{\Gamma_N} g_1 \cdot v \, ds.
\]
Therefore for all test-functions $v$ with $v(x) = 0$ on $\Gamma_D$, it holds

$$\int_{\Omega} \langle \nabla u, \nabla v \rangle \, dx + c \cdot \int_{\Omega} u \cdot v \, dx = \int_{\Omega} f \cdot v \, dx + \int_{\Gamma_N} g_1 \cdot v \, ds.$$

Notice now three things:

1. We have not imposed yet the Dirichlet boundary condition ($u = g_0$ on $\Gamma_D$). Nevertheless, we have imposed a similar one to the function $v$, but in a homogeneous way ($v = 0$ on $\Gamma_D$).

2. As written now, data ($f$ and $g_1$) are in the right-hand side and coefficients of the equation (the only one we have is $c$) are in the left-hand side.

3. The expression on the left-hand side is linear in both $u$ and $v$. It is a **bilinear form** of the variables $u$ and $v$. The expression on the right-hand side is linear in $v$. 
Without specifying spaces where \( u \) and \( v \) are, the weak formulation can be written as follows:

\[
\int_{\Omega} \langle \nabla u, \nabla v \rangle \, dx + c \cdot \int_{\Omega} u \cdot v \, dx = \int_{\Omega} f \cdot v \, dx + \int_{\Gamma_N} g_1 \cdot v \, ds,
\]

\[
u(x) = g_0(x) \quad x \in \Gamma_D,
\]

for all test-functions \( v \) with \( v(x) = 0 \) on \( \Gamma_D \).

Note how the two boundary conditions appear in very different places of this formulation:

- The Dirichlet condition (given displacements) is imposed apart from the formulation and involves imposing it homogeneously to the test-function \( v \). It is called an **essential boundary condition**.

- The Neumann condition (given normal stresses) appears inside the formulation. It is called a **natural boundary condition**.
What does essential and natural mean?

Being *essential* or *natural* is not inherently tied to the boundary condition: it is related to the role of the boundary condition in the formulation.

So when you hear (or say) essential boundary condition, you mean a boundary condition that is imposed apart from the formulation, whereas a natural boundary condition appears inside the formulation.

For this weak formulation of a second order elliptic equation we have:

\[
\begin{align*}
\text{Dirichlet} & = \text{essential}, \\
\text{Neumann} & = \text{natural}.
\end{align*}
\]
What are the test-functions $v$?

At this point, you might (you should) be wondering what is $v$ in the formulation. In the jargon of weak formulations, $v$ is called a **test-function**. It tests the equation that is satisfied by $u$.

The main idea is that instead of looking at the equation as something satisfied point-by-point in the domain —, you have an averaged version of the equation. Then $v$ plays the role of a weight function, something you use to average the equation.

In many contexts (books on mechanics, engineering or physics) $v$ is called a virtual displacement (or virtual work, or virtual whatever is pertinent), emphasizing the fact that $v$ is not the unknown of the system, but something that only exists virtually to write down the problem. The weak formulation is, in that context, a principle of virtual displacements (principle of virtual work, etc).
Short Exercise Session
Classroom Exercise: obtaining weak formulations

- Derive the weak form of the Poisson equation in 2D:
  \[- \Delta u(x, y) = f(x, y).\]

- Derive the weak form of the Poisson equation in 1D:
  \[-u_{x,x}(x) = f(x).\]

- Derive the weak form of the heat equation in 1D:
  \[u_t(t, x) - u_{x,x}(t, x) = 0.\]

Caveat: Take care about a proper definition of the test-functions.
The Mathematics Behind
\[
\int_{\Omega} \langle \nabla u, \nabla v \rangle \, dx + c \cdot \int_{\Omega} u \cdot v \, dx = \int_{\Omega} f \cdot v \, dx + \int_{\Gamma_N} g_1 \cdot v \, ds.
\]

We have reached a point where we should be a little more specific on where we are looking for \( u \) and where \( v \) belongs. The first space we need is the space of square-integrable functions

\[
L^2(\Omega) := \left\{ f : \Omega \to \mathbb{R} : \int_{\Omega} |f(x)|^2 \, dx < \infty \right\}.
\]

A fully precise definition of this space requires either the introduction of the Lebesgue integral or applying some limiting ideas. If you know what this is all about, good for you! If you don’t, go on: for most functions you know you will always be able to check whether they belong to this space or not by computing or estimating the integral and seeing if it is finite or not.
The second space is one of the wide family of Sobolev spaces:

\[ H^1(\Omega) := \left\{ u \in L^2(\Omega) : u_{x_1}, u_{x_2} \in L^2(\Omega) \right\} , \]

i.e. \( u \) and its first partial derivatives are square integrable. There is a norm related to this space

\[
\|u\|_{H^1(\Omega)} := \left( \int_\Omega |u|^2 \, dx + \int_\Omega \|\nabla u\|^2 \, dx \right)^{1/2} = \left( \int_\Omega |u|^2 + |u_{x_1}|^2 + |u_{x_2}|^2 \, dx \right)^{1/2} .
\]
Sometimes this norm

$$\|u\|_{H^1(\Omega)} = \left( \int_\Omega |u|^2 \, dx + \int_\Omega \|\nabla u\|^2 \, dx \right)^{1/2}$$

is called the energy norm and functions that have this norm finite (that is, functions in $H^1(\Omega)$) are called functions of finite energy. The concept of energy is however related to the particular problem, so it’s better to get used to have the space and its norm clearly written down and think of belonging to this space as a type of admissibility condition.

A particular subset of this space will be of interest for us:

$$H^1_{\Gamma_D}(\Omega) \ := \ \{ \ v \in H^1(\Omega) \ : \ v(x) = 0, \ \forall \ x \in \Gamma_D \ \}.$$

Note that this test-function space $H^1_{\Gamma_D}(\Omega)$ is a subspace of $H^1(\Omega)$, that is, linear combinations of elements of $H^1_{\Gamma_D}(\Omega)$ belong to the same space.
How do we make sense of all these definitions? (1/2)

An even half-trained mathematician should be wondering what do we mean by the partial derivatives in the definition of $H^1(\Omega)$, since one cannot think of taking the gradient of an arbitrary function of $L^2(\Omega)$, or at least to taking the gradient and finding something reasonable.

What we mean by restriction to $\Gamma_D$ in the definition of $H^1_{\Gamma_D}(\Omega)$ is not clear either, since elements or $L^2(\Omega)$ are not really functions, but classes of functions, where values of the function on particular points or even on lines are not relevant.

To make this completely precise there are several ways:

1. Define a weak derivative for elements of $L^2(\Omega)$ and what we understand by saying that that derivative is again in $L^2(\Omega)$. Then you move to give a meaning to that restriction of a function in $H^1(\Omega)$ to one part of its boundary.
2. Go the whole nine yards and take time to browse a book on distribution theory and Sobolev spaces. It takes a while but you end up with a pretty good intuition of what this all is about.

3. Take the short way. You first consider the space of functions

\[ C^1(\Omega) := \{ u \in C(\Omega) : u_{x_1}, u_{x_2} \in C(\Omega) \} , \]

which is simple to define, and then you close it with the norm \( \| \cdot \|_{H^1(\Omega)} \). To do that you have to know what closing or completing a space is (it’s something similar to what you do to define real numbers from rational numbers). Then you have to prove that restricting to \( \Gamma_D \) still makes sense after this completion procedure.

My recommendation at this point is to simply go on. Nevertheless, if you keep on doing research related to finite elements, you should really know something more about this.
The Complete Picture of the Weak Form
With the spaces defined above we can finally write our problem in a proper and fully rigorous way: Find \( u \in H^1(\Omega) \), such that

\[
\int_{\Omega} \langle \nabla u , \nabla v \rangle \, dx + c \cdot \int_{\Omega} u \cdot v \, dx = \int_{\Omega} f \cdot v \, dx + \int_{\Gamma_N} g_1 \cdot v \, ds,
\]

\[
u(x) = g_0(x) \quad x \in \Gamma_D,
\]

for all test-functions \( v \in H^1_{\Gamma_D}(\Omega) \).

Let us recall that the condition on the general test function \( v \in H^1_{\Gamma_D}(\Omega) \) is the same as

\[
v \in H^1(\Omega), \quad \text{such that } v = 0 \text{ on } \Gamma_D,
\]

that is, \( v \) is in the same space as the unknown \( u \) but satisfies a homogeneous version of the Dirichlet/essential boundary condition.
The data $f$, $g_1$ and $g_0$ are in the following spaces

$$f \in L^2(\Omega), \quad g_1 \in L^2(\Gamma_N), \quad g_0 \in H^{1/2}(\Gamma_D).$$

We have already spoken of the first of these spaces. The space $L^2(\Gamma_N)$ is essentially the same idea, with line integrals on $\Gamma_N$ instead of domain integrals on $\Omega$.

The last space looks more mysterious: it is simply the space of restrictions to $\Gamma_D$ of functions of $H^1(\Omega)$, that is, $g_0 \in H^{1/2}(\Gamma_D)$ means that there exists at least a function $u_0 \in H^1(\Omega)$ such that $u_0 = g_0$ on $\Gamma_D$. 

In fact, all other functions satisfying this condition

that there is a function \( u_0 \in H^1(\Omega) \) such that \( u_0 = g_0 \) on \( \Gamma_D \)

(in particular our solution \( u \)) belong to

\[
\begin{align*}
 u_0 + H^1_{\Gamma_D}(\Omega) &= \{ u_0 + v : v \in H^1_{\Gamma_D}(\Omega) \} \\
&= \{ w \in H^1(\Omega) : w(x) = g_0(x) \forall x \in \Gamma_D \}
\end{align*}
\]

(can you see why?). Unlike \( H^1_{\Gamma_D}(\Omega) \), the set \( u_0 + H^1_{\Gamma_D}(\Omega) \) is not a subspace of \( H^1(\Omega) \). The only exception is the trivial case, when \( g_0 = 0 \), since then then set becomes \( H^1_{\Gamma_D}(\Omega) \).

That \( g_0 \) belongs to \( H^{1/2}(\Gamma_D) \) means simply that we are not looking for the solution on the empty set. I cannot give you here a simple and convincing explanation on the name of this space. Sorry for that.
Short Exercise Session
Classroom Exercise: function spaces for weak formulations

- Give the correct spaces for the functions involved in the weak form of the Poisson equation in 2D:
  \[-\Delta u(x, y) = f(x, y).\]

- Give the correct spaces for the functions involved in the weak form of the Poisson equation in 1D:
  \[-u_{x,x}(x) = f(x).\]

- Give the correct spaces for the functions involved in the weak form of the heat equation in 1D:
  \[u_t(t, x) - u_{x,x}(t, x) = 0.\]
The Discretization’s Prelude: Affine Functions on a Triangle & Triangulations
First of all, let us think for a moment about linear and affine functions. An affine function of two variables is the same as a polynomial function of degree at most one

\[ p(x_1, x_2) = a_0 + a_1 x_1 + a_2 x_2. \]

The set of these functions is denoted \( \mathbb{P}_1 \). An affine function is uniquely determined by its values on three different non-aligned points, that is, on the vertices of a (non-degenerate) triangle.

A triangle \( K \) and its three vertices.
Thus, a function

\[ p \in \mathbb{P}_1 := \{ a_0 + a_1 x_1 + a_2 x_2 : a_0, a_1, a_2 \in \mathbb{R} \}, \]

can be determined either

- from its three defining coefficients \((a_0, a_1, a_2)\), or
- from its values on the three vertices of a triangle \(K\).

Both possibilities state that the space \(\mathbb{P}_1\) is a vector space of dimension three. While the first choice (coefficients) gives us a simple expression of the function, the second is more useful for many tasks, in particular for drawing the function.

The three values of the function on the vertices will be called the **local degrees of freedom**.
There is another important property that will be extremely useful in the sequel:

the value of \( p \in \mathbb{P}_1 \) on the edge that joins two vertices of the triangle depends only on the values of \( p \) on these two vertices.

In other words, the value of \( p \in \mathbb{P}_1 \) on an edge is uniquely determined by the degrees of freedom associated to the edge, namely, the values of \( p \) on the two vertices that lie on that edge.
So far we have functions on a single triangle. Now we go for partitions of the domain into triangles. A **triangulation** of \( \Omega \) is a subdivision of this domain into triangles.

Triangles must cover all – but no more and must fulfil the following rule:

> if two triangles have some intersection, it is either on common vertex or a common full edge. In particular, two different triangles do not overlap.

There is another rule, related to the partition of \( \Gamma \) into \( \Gamma_D \) and \( \Gamma_N \):

> the triangulation must respect the partition of the boundary into Dirichlet and Neumann boundaries.

I.e. that an edge of a triangle that lies on \( \Gamma \) cannot be part Dirichlet and part Neumann. Thus if there is a transition from Dirichlet- to Neumann-boundaries, there must be a vertex of a triangle in that transition point.
Examples of forbidden triangulations

Situations not admitted in triangulations. In the second one we see the appearance of what is called a hanging node.

In the third situation, we see a forbidden transition of Dirichlet- to Neumann-boundary conditions happening inside an edge.
The fineness/coarseness of a triangulation is measured by the length of the longest edge of all triangles.

Example of a proper triangulation.

The set of the triangles (that is, the list thereof) will be generally denoted $\mathcal{T}_h$. The subindex $h$ makes reference to the diameter of the triangulation, defined as the length of the longest edge of all triangles, that is, the longest distance between vertices of the triangulation.