Exercise Session
Page 175, exercise 10 a
Use a divided differences table to construct Newton’s interpolation polynomial for the following data:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>7</td>
<td>11</td>
<td>28</td>
<td>63</td>
</tr>
</tbody>
</table>

Page 175, exercise 10 b
Without simplifying the Newton interpolation polynomial obtained above, write this polynomial in nested form for easy evaluation.
Page 176, exercise 22

Find a polynomial of least degree that takes these values:

<table>
<thead>
<tr>
<th>$x$</th>
<th>1.73</th>
<th>1.82</th>
<th>2.61</th>
<th>5.22</th>
<th>8.26</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = y$</td>
<td>0</td>
<td>0</td>
<td>7.8</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

*Hint:* Rearrange the table so that the non-zero value of $f(x)$ is the last entry, or think of some better way.
Introduction & Today's Scope
When a function $f$ is approximated on an interval $[a, b]$ by means of an interpolating polynomial $p$, the discrepancy between $f$ and $p$ will (theoretically) be 0 at each node of the interpolation.

A natural expectation is that the function $f$ is well approximated at all intermediate points, too, and that as the number of points increases, this agreement will become better and better.

Unfortunately, as we will see this does not have to be the case even for smooth functions $f$.

(Of course, if the function being approximated is not required to be continuous, then there may be no agreement at all between $p(x)$ and $f(x)$ except at the nodes.)
Interpolation of $x \mapsto (1 + x^2)^{-1}$ on an equidistant grid with 5 nodes.
Interpolation of $x \mapsto (1 + x^2)^{-1}$ on a equidistant grid with 7 nodes.
Interpolation of $x \mapsto (1 + x^2)^{-1}$ on a equidistant grid with 9 nodes.
Interpolation of \( x \mapsto (1 + x^2)^{-1} \) on a equidistant grid with 11 nodes.
Interpolation of $x \mapsto (1 + x^2)^{-1}$ on a equidistant grid with 13 nodes.
Interpolation of $x \mapsto (1 + x^2)^{-1}$ on an equidistant grid with 15 nodes.
Interpolation of \( x \mapsto (1 + x^2)^{-1} \) on an equidistant grid with 17 nodes.
Interpolation of $x \mapsto (1 + x^2)^{-1}$ on a equidistant grid with 21 nodes.
Interpolation of \( x \mapsto (1 + x^2)^{-1} \) on a equidistant grid with 41 nodes.
Interpolation of $x \mapsto (1 + x^2)^{-1}$ on an equidistant grid with 81 nodes.
Today, we will focus on errors in polynomial interpolation

Today’s topics:

■ Sources of errors in polynomial interpolation
■ The role of Chebyshev nodes
■ Theorems on interpolation errors (just the results)
■ Condition of polynomial interpolation

Corresponding textbook chapters: 4.1
Illustrative Examples: Errors in Polynomial Interpolation
Example

Consider the five nodes

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>3</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>8</td>
<td>12</td>
<td>2</td>
<td>6</td>
<td>0</td>
</tr>
</tbody>
</table>

and construct and plot the interpolation polynomial using the two outermost points. Repeat this process by adding one additional point at a time until all the points are included. What conclusions can you draw?

Before we have a look at the graphs, can one of draw his/her expectation of the interpolating polynomial through these five nodes on the blackboard?
Graphs of the interpolating polynomials over the data points:
Graphs of the interpolating polynomials over the data points:
Poor fitting polynomials (2/3)

Graphs of the interpolating polynomials over the data points:
Graphs of the interpolating polynomials over the data points:
Graphs of the interpolating polynomials over the data points:
Example [Discussion of the Results]

We were hoping for a smooth curve going through the nodes without wide fluctuations, but this did not happen. It may seem counterintuitive, but as we added more points, the situation became worse instead of better!

The reason for this comes from the nature of high-degree polynomials. A polynomial of degree $n$ has $n$ zeros. If all these zeros are real, then the curve crosses the $x$-axis $n$ times. The resulting curve must make many turns for this to happen, resulting in wild oscillations.

A remedy is to use another interpolation method, e.g. (natural) spline interpolation. (Think of the spline functions as “bamboo rods” that you fit around the nodes. Due to the elastic characteristics a energy/ tension optimal curve results naturally.)
A function \( s : [a, b] \rightarrow \mathbb{R} \) is called spline of order \( m \) or of degree \( m - 1 \), if the following holds:

- \( s(x) = p_i(x) \) on \([x_i, x_{i+1}]\) with \( p_i \in \mathcal{P}_{m-1}(x_i, x_{i+1})\), 
  \( i = 0, 1, \ldots, n - 1 \),
- \( s \in C^{m-2}([a, b]) \).

Thus, between two neighboring nodes the spline \( s \) is a polynomial of degree \( m - 1 \) and globally (particularly in the knots themselves) it is \( m - 2 \) times continuously differentiable.

Thus, we can think of a spline \( s \) as a piecewise polynomial.

E.g.: for \( m = 4 \) we obtain cubic splines (piecewise cubic, globally twice continuously differentiable).

About the name: The English word spline describes an elastic wooden slat used in shipbuilding.
Example

As a pathological example, consider the **Dirichlet function**

\[ f(x) := \begin{cases} 
0 & \text{if } x \text{ is rational} \\
1 & \text{if } x \text{ is irrational} 
\end{cases}. \]

If we choose nodes that are rational numbers, then \( p(x) \equiv 0 \) and \( f(x) - p(x) = 0 \) for all rational values of \( x \). Though, \( f(x) - p(x) = 1 \) for all irrational values of \( x \).

However, if the function \( f \) is well behaved, we can we not assume that the differences \( |f(x) - p(x)| \) are small when the number of interpolating nodes is large?

The answer is still **no**, even for functions that possess continuous derivatives of all orders on the interval!
Example

A specific example of the remarkable phenomenon that an increasing number of nodes for the polynomial interpolation does not decrease the interpolation accuracy is provided by the **Runge function**

\[ f(x) = \frac{1}{1 + x^2}, \quad x \in [-5, 5]. \]

Let \( p_n \) be the polynomial that interpolates this function at \( n + 1 \) equally spaced points on the interval \([-5, 5]\), including the endpoints. Then,

\[ \lim_{n \to \infty} \max_{-5 \leq x \leq 5} |f(x) - p(x)| = \infty. \]

Thus, the effect of requiring the agreement of \( f \) and \( p_n \) at more and more points is to increase the error at non-nodal points, and the error actually increases beyond all bounds!
Polynomial interpolation of the Runge function with 41 equally spaced nodes:
Example [Discussion of the Results]

The moral of this example is that polynomial interpolation of high degree with many nodes is a risky operation; the resulting polynomials may be very unsatisfactory as representations of functions unless the set of nodes is chosen with great care.

In a more advanced study of this topic, it would be shown that the divergence of the polynomials can often be ascribed to the fact that the nodes equally spaced. Again, contrary to intuition, equally distributed nodes are usually a very poor choice in interpolation. A much better choice for $n + 1$ nodes in $[-1, 1]$ is the set of Chebysev nodes:

$$x_i = \cos \left( \left( \frac{2i + 1}{2n + 2} \right) \pi \right), \quad \text{for } i = 0, 1, 2, \ldots, n.$$
The Chebysev nodes are obtained by taking equally spaced points on the unit circle and projecting them onto the horizontal axis:

\[
n \arccos x = \text{odd} \cdot \pi/2
\]

\[
x = \cos \frac{\text{odd} \cdot \pi}{2n}.
\]

**Figure 3.10 Location of zeros of the Chebyshev polynomial.** The roots are the \( x \)-coordinates of evenly spaced points around the circle. (a) degree 5 (b) degree 15 (c) degree 25.
The Chebysev nodes (2/2) ...

Defined on $[-1, 1]$ the **Chebysev nodes**

$$x_i = \cos \left( \left( \frac{2i + 1}{2n + 2} \right) \pi \right), \quad \text{for } i = 0, 1, 2, \ldots, n.$$ 

can be placed upon an arbitrary interval $[a, b]$ by applying a linear mapping that results in

$$x_i = \frac{1}{2} (a + b) + \frac{1}{2} (b - a) \cos \left( \left( \frac{2i + 1}{2n + 2} \right) \pi \right), \quad \text{for } i = 0, 1, 2, \ldots, n$$

as the Chebysev nodes on $[a, b]$. Notice that these nodes are numbered from right to left. Since the theory does not depend any particular ordering of the nodes this is not troublesome.
Interpolation of the Runge function on a Chebyshev grid with 5 nodes.
Interpolation of the Runge function on a Chebysev grid with 7 nodes.
... and their application to the Runge function

Interpolation of the Runge function on a Chebysev grid with 9 nodes.
... and their application to the Runge function

Interpolation of the Runge function on a Chebysev grid with 11 nodes.
Interpolation of the Runge function on a Chebysev grid with 13 nodes.
... and their application to the Runge function

Interpolation of the Runge function on a Chebysev grid with 15 nodes.
Interpolation of the Runge function on a Chebysev grid with 17 nodes.
... and their application to the Runge function

Interpolation of the Runge function on a Chebysev grid with 21 nodes.

![Graph of the Runge function on a Chebysev grid with 21 nodes.](image)
Interpolation of the Runge function on a Chebysev grid with 41 nodes.
Interpolation of the Runge function on a Chebysev grid with 81 nodes.
Theorems on Interpolation Errors
Some facts about divided differences

**Theorem (Divided Differences & Derivatives)**

Let $f^{(n)}$ be continuous on $[a, b]$ and $x_0, x_1, \ldots, x_n$ be any $n + 1$ distinct points in $[a, b]$. Then, for some $\xi \in (a, b)$ it holds that

$$f[x_0, x_1, \ldots, x_n] = \frac{1}{n!} f^{(n)}(\xi).$$

**Divided Differences Corollary**

If $f$ is a polynomial of degree $n$, then all of the divided differences $f[x_0, x_1, \ldots, x_i]$ are zero for $i \geq n + 1$. 
Example

Is there a cubic polynomial that takes these values?

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>−2</th>
<th>0</th>
<th>3</th>
<th>−1</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = y$</td>
<td>−2</td>
<td>−56</td>
<td>−2</td>
<td>4</td>
<td>−16</td>
<td>376</td>
</tr>
</tbody>
</table>

If such a cubic polynomial exists, its fourth order divided differences $f[, , , , ]$ would all be zero. Thus, we form a divided difference table to check this possibility:
Existence of specific order interpolating polynomials

Example [cont.]

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>18</td>
<td>-9</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>-2</td>
<td>-56</td>
<td>-9</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>-2</td>
<td>5</td>
<td>-3</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>5</td>
<td>-3</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>-16</td>
<td>11</td>
<td>49</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>376</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example [cont.]

The data can thus be represented by a cubic polynomial because the fourth order divided differences \( f[ , , , , ] \) are zero. From the Newton form of the interpolation formula, this unique polynomial is

\[
p_3(x) = -2 + 18(x - 1) - 9(x - 1)(x + 2) + 2(x - 1)(x + 2)x.
\]
Theorem 1 (Interpolation Error)

Let $p$ be a polynomial of degree at most $n$ that interpolates $f \in C^0$ at the $n+1$ distinct nodes $x_0, x_1, \ldots, x_n$. Then, for any $x \in \mathbb{R}$ that is not a node

$$f(x) - p(x) = f[x_0, x_1, \ldots, x_n, x] \cdot \prod_{i=0}^{n} (x - x_i).$$

Let $p$ be a polynomial of degree at most $n$ that interpolates $f \in C^{n+1}$ at the $n+1$ distinct nodes $x_0, x_1, \ldots, x_n$ belonging to the interval $[a, b]$. Then, for any $x \in [a, b]$, there is a $\xi \in (a, b)$ for which

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \cdot \prod_{i=0}^{n} (x - x_i).$$
The divided difference part of the theorem on interpolation errors is a straightforward interpretation of the algorithm that led to Newton form of the interpolation polynomial:

Let \( t \) be any point, other than a node, where \( f(t) \) is defined. Let \( q \) be a polynomial of degree \( \leq n + 1 \) that interpolates \( f \) at \( x_0, x_1, \ldots, x_n, \) and \( t \). By the Newton form of the interpolation formula, we have

\[
q(x) = p(x) + f[x_0, x_1, \ldots, x_n, t] \cdot \prod_{i=0}^{n} (x - x_i) .
\]

Since \( q(t) = f(t) \), this yields at once

\[
q(t) = p(t) + f[x_0, x_1, \ldots, x_n, t] \cdot \prod_{i=0}^{n} (x - x_i) .
\]
The upper bound lemma

A special case of node distribution that often arises is the one in which the interpolation nodes are equally spaced. In this case, we have:

Upper Bound Lemma

Let \( x_i = a + ih \) for \( i = 0, 1, \ldots, n \) and \( h = (b-a)/n \). Then for any \( x \in [a, b] \) it holds that

\[
\prod_{i=0}^{n} |x - x_i| \leq \frac{1}{4} \cdot h^{n+1} \cdot n!
\]
The 2nd part of the theorem on interpolation errors together with the upper bound lemma implies:

**Theorem 2 (Interpolation Error)**

Let \( f \in C^{n+1}[a, b] \) such that \(|f^{(n+1)}(x)| \leq M\). Let \( p \) be a polynomial of degree at most \( n \) that interpolates \( f \) at the \( n + 1 \) distinct and equally spaced nodes \( a = x_0, x_1, \ldots, x_n = b \) belonging to the interval \([a, b]\). Then, on \([a, b]\), it holds that

\[
|f(x) - p(x)| \leq \frac{1}{4(n + 1)} M h^{n+1},
\]

where \( h = (b - a)/n \) is the spacing between the nodes.
Example

Assess the error if \( \sin(x) \) is replaced by an interpolation polynomial that has 10 equally spaced nodes in \([0, 1.6875]\).

We use the 2nd theorem on interpolation errors, taking \( f(x) = \sin(x) \), \( n = 9 \), \( a = 0 \) and \( b = 1.6875 \). Since \( f^{(10)}(x) = -\sin(x) \), \( |f^{(10)}(x)| \leq 1 \) and we can apply the 2nd theorem on interpolation errors with \( M = 1 \). The result is

\[
|\sin(x) - p(x)| \leq 1.34 \cdot 10^{-9}.
\]

Thus, the interpolation polynomial \( p(x) \) that has 10 equally spaced nodes on \([0, 1.6875]\) represents \( \sin(x) \) on this interval to at least 8 decimal digits of accuracy.

In fact a careful check on a computer would reveal that the polynomial is accurate to even more decimal places. (Why? – think Taylor!)
Classroom Problem

Suppose $\cos(x)$ is to be approximated by an interpolating polynomial of degree $n$, using $n + 1$ equally spaced nodes in the interval $[0, 1]$.

- How accurate is the approximation? (Express your answer in terms of $n$.)
- How accurate is the approximation for $n = 9$?
- For what values of $n$ is the error less than $10^{-7}$?
With some effort one can show the following:

**Theorem 3 (Interpolation Error)**

Let \( p \) be a polynomial of degree at most \( n \) that interpolates \( f \in C^{n+1} \) at the \( n + 1 \) distinct Chebysev nodes

\[
x_i = \cos \left( \left( \frac{2i + 1}{2n + 2} \right) \pi \right), \quad \text{for } i = 0, 1, 2, \ldots, n.
\]

on the interval \([-1, 1]\). Then, it holds that

\[
|f(x) - p(x)| \leq \frac{1}{2^n (n + 1)!} \max_{-1 \leq \xi \leq 1} \left| f^{(n+1)}(\xi) \right|.
\]

Moreover, this is the best upper bound we can achieve by varying the choice of the \( x_i \).
Condition of Polynomial Interpolation
Recall, that condition in our setting describes the sensitivity of the interpolation problem to changes in the input data, i.e. the table

\[
\begin{array}{c|c|c|c|c|c}
  x & x_0 & x_1 & x_2 & \cdots & x_n \\
  \hline
  f(x) = y & y_0 & y_1 & y_2 & \cdots & y_n
\end{array}
\]

Therefore, we can think of condition numbers depending on the input data with respect to which the sensitivity is to be examined:

- It is easiest to describe the sensitivity of the interpolation polynomial \( p(x) \) on \([a, b] \) regarding variations in its points of evaluation \( x \). When we think of a Taylor expansion of \( p(x + h) \), for a small deviation \( h \), these variations are described by \( p'(x) \). Hence are bounded on the compact set \([a, b] \) in the case of polynomials. Though, they can become very big, particularly if \( p(x) \) has a high degree.

- In practice, the sensitivity regarding variations in the nodes and especially variations of the supporting values \( y_i \) is more important.
In practice, the sensitivity regarding variations in the nodes and especially variations of the supporting values \( y_i \) is important as these represent the actually measured data.

From the Lagrange form

\[
y = p_n(x) = \sum_{i=0}^{n} y_i \cdot l_i(x)
\]

it immediately follows that

\[
\frac{\partial y}{\partial y_i} = l_i(x).
\]

Thus, the size of the Lagrange polynomials on \([a, b]\) determines this condition.

To get a feeling for the order of magnitude the values of the Lagrange polynomials \( l_i(x) \) can reach, we will discuss this via an illustrative example.
On the order of magnitude the values of the Lagrange polynomials can reach

**Example**

Give a sufficiently good lower bound for $|l_{20}(x)|$ on $(0, 1)$ for an arbitrary interpolation problem with the 40 nodes $x_k = k$, with $k = 0, 1, \ldots, 40 = n$.

We get the following estimation for the value of $|l_{20}|$ in $(0, 1)$:

$$|l_{20}(x)| = \left| \prod_{k=20}^{n} \frac{x - x_k}{x_{20} - x_k} \right| = \frac{x}{20} \frac{1 - x}{19} \prod_{k=21}^{40} \frac{k - x}{k - 20} \prod_{k=2}^{19} \frac{k - x}{20 - k} \geq \frac{x - x^2}{380} \prod_{k=2}^{40} \frac{k - 1}{k - 20} \prod_{k=21}^{19} \frac{k - 1}{k - 20} = \frac{x - x^2}{380} \frac{18!}{18!} \frac{39!}{20!} \geq 1.8 \cdot 10^8 \cdot x \cdot (1 - x).$$

In particular we get $|l_{20}(0.5)| \geq 4.5 \cdot 10^7$. 
On the order of magnitude the values of the Lagrange polynomials can reach

Example [Discussion of the Result]

- This is an fundamental result: Small errors in internal supporting values are dramatically increased at the borders of the examined interval by polynomial interpolation.

- For large numbers of nodes $n$ (7 or 8 and up), polynomial interpolation with equidistant nodes is extremely ill-conditioned and therefore basically useless.

- For this reason, better methods must be found ...
Summary & Outlook
The Runge function $f(x) = (1 + x^2)^{-1}$ on the interval $[-5, 5]$ shows that high-degree polynomial interpolation and uniform spacing of nodes may not be satisfactory. The Chebysev nodes are a better choice. For the interval $[a, b]$ they are given as

$$x_i = \frac{1}{2}(a + b) + \frac{1}{2}(b - a) \cos \left( \left( \frac{2i + 1}{2n + 2} \right) \pi \right), \quad \text{for } i = 0, 1, 2, \ldots, n.$$  

There is a relationship between divided differences and derivatives, namely:

$$f[x_0, x_1, \ldots, x_n] = \frac{1}{n!} f^{(n)}(\xi),$$

where $x_0, x_1, \ldots, x_n$ be any $n + 1$ distinct points in $[a, b]$, and $\xi \in (a, b)$ be appropriately chosen.

If $f$ is a polynomial of degree $n$, then all of the divided differences $f[x_0, x_1, \ldots, x_i]$ are zero for $i \geq n + 1$. 


Expressions for errors in polynomial interpolation are

\[ f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i), \]

\[ f(x) - p(x) = f[x_0, x_1, \ldots, x_n, x] \prod_{i=0}^{n} (x - x_i). \]

For \( n + 1 \) equally spaced nodes an upper bound on the error is given by

\[ |f(x) - p(x)| \leq \frac{M}{4(n+1)} \left( \frac{b - a}{n} \right)^{n+1}, \]

where \( M \) is an upper bound on \( |f^{(n+1)}(x)| \) for \( a \leq x \leq b \).

Moreover, for large numbers of nodes polynomial interpolation with equidistant nodes is extremely ill-conditioned and small errors in internal supporting values are dramatically increased at the borders of the examined interval.
Please, prepare these short exercises for the next lecture:

1. **Page 186, exercise 1**
   Use a divided difference table to show that the following data can be represented by a polynomial of degree 3:

<table>
<thead>
<tr>
<th>$x$</th>
<th>−2</th>
<th>−1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = y$</td>
<td>1</td>
<td>4</td>
<td>11</td>
<td>16</td>
<td>13</td>
<td>−4</td>
</tr>
</tbody>
</table>

2. **Page 186, exercise 10**
   Let the function $\ln(x)$ be approximated by an interpolation polynomial of degree 9 with 10 equally spaced nodes uniformly distributed in the interval $[1, 2]$. What can be placed on the error?
Please, prepare these short exercises for the next lecture:

3. **Page 186, exercise 10 (reformulated)**
   Let the function $\ln(x)$ be approximated by an interpolation polynomial of degree 9 with 10 Chebysev nodes on the interval $[1, 3]$. What can be placed on the error?