Exercise Session
Page 174, exercise 4
Verify that the polynomials

\[ p(x) = 5x^3 - 27x^2 + 45x - 21 \]
\[ q(x) = x^4 - 5x^3 + 8x^2 - 5x + 4 \]

interpolate the data

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = y )</td>
<td>2</td>
<td>1</td>
<td>6</td>
<td>47</td>
</tr>
</tbody>
</table>

and explain why this does not violate the uniqueness part of the existence and uniqueness theorem on polynomial interpolation.
We have

<table>
<thead>
<tr>
<th>$x$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = y$</td>
<td>$2$</td>
<td>$1$</td>
<td>$6$</td>
<td>$47$</td>
</tr>
<tr>
<td>$p(x)$</td>
<td>$2$</td>
<td>$1$</td>
<td>$6$</td>
<td>$47$</td>
</tr>
<tr>
<td>$q(x)$</td>
<td>$2$</td>
<td>$1$</td>
<td>$6$</td>
<td>$47$</td>
</tr>
</tbody>
</table>

and thus

$$p(x) = 5x^3 - 27x^2 + 45x - 21$$

and

$$q(x) = x^4 - 5x^3 + 8x^2 - 5x + 4$$

are both interpolating polynomials. Though, their degree differs and thus the uniqueness part is not violated.
Reviewing the highlights from last time

Given the table

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = y$</td>
<td>7</td>
<td>11</td>
<td>28</td>
<td>63</td>
</tr>
</tbody>
</table>

- **Page 174, exercise 1**
  Use the Lagrange interpolation process to obtain a polynomial of least degree that interpolates the above table.

- **Page 174, exercise 1 (reformulated)**
  Use the Newton interpolation process to obtain a polynomial of least degree that interpolates the above table.
Today’s topics:

- Calculating the coefficients for Newton’s interpolation ansatz using divided differences
- Interpolation with monomials and the Vandermode matrix

Corresponding textbook chapters: 4.1
Calculating the Newton Coefficients Using Divided Differences
We now turn to the problem of determining the coefficients \(a_0, a_1, \ldots, a_n\) of the Newton interpolation polynomial

\[
p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots + a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1}),
\]

efficiently. Again, we start with a table of values of a function \(f\):

<table>
<thead>
<tr>
<th>(x)</th>
<th>(x_0)</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(\ldots)</th>
<th>(x_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>(f(x_0))</td>
<td>(f(x_1))</td>
<td>(f(x_2))</td>
<td>(\ldots)</td>
<td>(f(x_n))</td>
</tr>
</tbody>
</table>

The points \(x_0, x_1, x_2, \ldots, x_n\) are assumed to be distinct, but no further assumption is made about their positions on the real line. We already know that for each \(n = 0, 1, \ldots\) there exists an unique polynomial \(p_n\) such that

- the degree of \(p_n\) is at most \(n\), and
- \(p_n(x_i) = f(x_i)\) for all \(i = 0, 1, \ldots, n\).
A crucial observation about the Newton interpolation polynomial

$$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots + a_n(x - x_0)(x - x_1)\ldots(x - x_{n-1}),$$

is that its coefficients $a_0, a_1, \ldots, a_n$ do not depend on $n$.

In other words, $p_n$ is obtained from $p_{n-1}$ by adding one more term, without altering the coefficients already present in $p_{n-1}$ itself:

$$p_n(x) = p_{n-1}(x) + a_n(x - x_0)(x - x_1)\ldots(x - x_{n-1}).$$
A systematic way of determining the unknown coefficients $a_0, a_1, \ldots, a_n$ from the table

<table>
<thead>
<tr>
<th>$x$</th>
<th>$x_0$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$\cdots$</th>
<th>$x_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>$f(x_0)$</td>
<td>$f(x_1)$</td>
<td>$f(x_2)$</td>
<td>$\cdots$</td>
<td>$f(x_n)$</td>
</tr>
</tbody>
</table>

is to set $x$ equal in turn to $x_0, x_1, \ldots, x_n$ during the construction of the Newton interpolation polynomial

\[
p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots + a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1}),
\]

i.e.

\[
p_0(x_0) = f(x_0) = a_0,
\]
\[
p_1(x_1) = f(x_1) = a_0 + a_1(x_1 - x_0),
\]
\[
p_2(x_2) = f(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1), \text{ etc.} \ldots
\]
The key idea in gaining the Newton coefficients (4/4)

The equations

\[
\begin{align*}
p_0(x_0) &= f(x_0) = a_0, \\
p_1(x_1) &= f(x_1) = a_0 + a_1(x_1 - x_0), \\
p_2(x_2) &= f(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1), \text{ etc.}
\end{align*}
\]

can be solved for the \(a_i\)'s in turn, starting with \(a_0\):

\[
\begin{align*}
a_0 &= f[x_0] := f(x_0) \\
a_1 &= f[x_0, x_1] := \frac{f(x_1) - a_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\
a_2 &= f[x_0, x_1, x_2] := \frac{f(x_2) - a_0 - a_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad \text{etc.}
\end{align*}
\]
Some remarks on Newton’s divided differences

The (bracket) notation

\[ a_k = f[x_0, x_1, \ldots, x_k] \]

indicates that \( a_k \) depends on the values of \( f \) at the nodes \( x_0, x_1, \ldots, x_n \).

As the values \( a_k \) are uniquely determined through the system of equations shown on the last slide we can view the identity \( a_k = f[x_0, x_1, \ldots, x_k] \) as definition of the quantity \( f[x_0, x_1, \ldots, x_k] \).

The quantity \( f[x_0, x_1, \ldots, x_k] \) is called divided difference of order \( k \) for \( f \).
Example: Determining divided differences

Example

Determine the quantities \( f[x_0] \), \( f[x_0, x_1] \) and \( f[x_0, x_1, x_2] \) based on the table

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>−4</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>3</td>
<td>13</td>
<td>−23</td>
</tr>
</tbody>
</table>

The determining systems of equations reads as

\[
\begin{align*}
p_0(x_0) &= f(x_0) = 3 = a_0, \\
p_1(x_1) &= f(x_1) = 13 = a_0 + a_1(x_1 - x_0) = a_0 + a_1(-5), \\
p_2(x_2) &= f(x_2) = -23 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) \\
&= a_0 + a_1(-1) + a_2(-1)(4). 
\end{align*}
\]
Example [cont.]

From

\[ 3 = a_0 , \]
\[ 13 = a_0 - 5a_1 , \]
\[ -23 = a_0 - a_1 - 4a_2 , \]

we can read of the coefficients of the Newton interpolation polynomial as

\[ a_0 = 3 = f[1] \]
\[ a_1 = -2 = f[1,-4] \]
\[ a_2 = 7 = f[1,-4,0] . \]
Newton’s form of the interpolating polynomial

With the notation of the divided differences, the **Newton form of the interpolating polynomial** takes the form

\[
p_n(x) = \sum_{i=0}^{n} a_i \prod_{j=0}^{i-1} (x - x_j)
\]

\[
= \sum_{i=0}^{n} f[x_0, x_1, \ldots, x_i] \prod_{j=0}^{i-1} (x - x_j),
\]

with the convention \( \prod_{j=0}^{-1} (x - x_j) := 1 \).

Notice that the coefficient of \( x^n \) in \( p_n \) is \( f[x_0, x_1, \ldots, x_n] \) because the term \( x^n \) occurs only in \( \prod_{j=0}^{n-1} (x - x_j) \).

Thus, if \( f \) is a polynomial of degree \( \leq n - 1 \), then \( f[x_0, x_1, \ldots, x_n] = 0 \) leading to a Newton interpolating polynomial of degree \( \leq n - 1 \), too.
The determining system of the coefficients (a.k.a. the divided differences)

\[ f(x_0) = a_0 , \]
\[ f(x_1) = a_0 + a_1(x_1 - x_0) , \]
\[ f(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) , \text{ etc.} \]

can be rewritten as

\[ f(x_k) = \sum_{i=0}^{k} a_i \prod_{j=0}^{i-1} (x_k - x_j) , \text{ for } 0 \leq k \leq n . \]

It is evident, that the computation of the coefficients can be performed recursively. E.g. \( a_1 \) can be computed once \( a_0 \) is known, \( a_2 \) can be computed once \( a_0 \) and \( a_1 \) are known and so on.
We rewrite \( f(x_k) = \sum_{i=0}^{k} a_i \prod_{j=0}^{i-1} (x_k - x_j) \) as

\[
f(x_k) = a_k \prod_{j=0}^{k-1} (x_k - x_j) + \sum_{i=0}^{k-1} a_i \prod_{j=0}^{i-1} (x_k - x_j),
\]

and get

\[
a_k = \frac{f(x_k) - \sum_{i=0}^{k-1} a_i \prod_{j=0}^{i-1} (x_k - x_j)}{\prod_{j=0}^{k-1} (x_k - x_j)}
\]

which allows a recursive computation of the coefficients \( a_i = f[x_0, x_1, \ldots, x_i] \).
Algorithm for computing the divided differences

Algorithm (Divided Differences)

1. Set $f[x_0] = f(x_0)$.
2. For $k = 1, 2, \ldots, n$ compute $f[x_0, x_1, \ldots, x_k]$ via

$$f[x_0, x_1, \ldots, x_k] = \frac{f(x_k) - \sum_{i=0}^{k-1} f[x_0, x_1, \ldots, x_i] \prod_{j=0}^{i-1} (x_k - x_j)}{\prod_{j=0}^{k-1} (x_k - x_j)}$$

The divided differences $f[x_0], f[x_0, x_1], \ldots, f[x_0, x_1, \ldots, x_n]$ can thus be computed at the cost of $\frac{1}{3}n(3n + 1)$ additions, $(n - 1)(n - 2)$ multiplications and $n$ divisions.
Example

Use our algorithm to write out the divided difference formulas for $f(x_0)$, $f(x_0, x_1)$, $f(x_0, x_1, x_2)$ and $f(x_0, x_1, x_2, x_3)$.

We have:

\[
\begin{align*}
  f[x_0] &= f(x_0) \\
  f[x_0, x_1] &= \frac{f(x_1) - f[x_0]}{x_1 - x_0} \\
  f[x_0, x_1, x_2] &= \frac{f(x_2) - f[x_0] - f[x_0, x_1](x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} \\
  f[x_0, x_1, x_2, x_3] &= \frac{f(x_3) - f[x_0] - f[x_0, x_1](x_3 - x_0) - f[x_0, x_1, x_2](x_3 - x_0)(x_3 - x_1)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}
\end{align*}
\]
Improving the Computation of the Divided Differences
We have just derived an algorithm that computes the divided differences \( f[x_0], f[x_0, x_1], \ldots, f[x_0, x_1, \ldots, x_n] \) at the cost of \( \frac{1}{3}n(3n + 1) \) additions, \((n - 1)(n - 2)\) multiplications and \( n \) divisions.

In this section, we will improve this result such that our new algorithm requires only

- \( n(n + 1) \) additions, \( \frac{1}{2}n(n + 1) \) divisions, and
- the storage of just three statements (!).

The tools that allow for the derivation of this algorithm are

- the recursive property of the divided differences, and
- the invariance theorem for divided differences.
The recursive property of the divided differences

**Theorem (Recursive Property of the Divided Differences)**

The divided differences obey the formula

\[
 f[x_0, x_1, \ldots, x_k] = \frac{f[x_1, x_2, \ldots, x_k] - f[x_0, x_1, \ldots, x_{k-1}]}{x_k - x_0}
\]

For instance, this allows the computation of \( f[x_0, x_1, x_2] \) via the scheme

\[
\begin{align*}
 f[x_0] & \quad \rightarrow \quad f[x_0, x_1] \\
 f[x_1] & \quad \rightarrow \quad f[x_0, x_1, x_2] \\
 f[x_2] & \quad \rightarrow \quad f[x_1, x_2]
\end{align*}
\]
Since $f[x_0, x_1, \ldots, x_k]$ was defined to be equal to the coefficient $a_k$ in the
Newton form of the interpolating polynomial

$$p_k(x) = \sum_{i=0}^{k} a_i \prod_{j=0}^{i-1} (x - x_j),$$

we say that $f[x_0, x_1, \ldots, x_k]$ is the coefficient of $x^k$ in the polynomial $p_k$ of
degree $\leq k$ which interpolates the function $f$ at the points $x_0, x_1, \ldots, x_k$.

Similarly, $f[x_1, x_2, \ldots, x_k]$ is the coefficient of $x^{k-1}$ in the polynomial $q$ of
degree $\leq k - 1$ which interpolates the function $f$ at the points $x_1, x_2, \ldots, x_k$.

And finally, we can interpret $f[x_0, x_1, \ldots, x_{k-1}]$ as the coefficient of $x^{k-1}$ in
the polynomial $p_{k-1}$ of degree $\leq k - 1$ which interpolates the function $f$ at
the points $x_0, x_1, \ldots, x_{k-1}$.
We have

- $f[x_0, x_1, \ldots, x_k]$ is the coefficient of $x^k$ in the polynomial $p_k$,
- $f[x_1, x_2, \ldots, x_k]$ is the coefficient of $x^{k-1}$ in the polynomial $q$, and
- $f[x_0, x_1, \ldots, x_{k-1}]$ is the coefficient of $x^{k-1}$ in the polynomial $p_{k-1}$.

These three polynomials $p_k$, $q$ and $p_{k-1}$ are related as follows:

\[
p_k(x) = q(x) + \frac{x - x_k}{x_k - x_0} (q(x) - p_{k-1}(x)).
\]

To establish this, we utilize the interpolation properties of $p_k$, $q$ and $p_{k-1}$.

First, we see that the right hand side is a polynomial of degree at most $k$. Next, we evaluate the right hand side at $x_i$, $i = 1, \ldots, k - 1$. This gives

\[
q(x_i) + \frac{x_i - x_k}{x_k - x_0} (q(x_i) - p_{k-1}(x_i)) = f(x_i) + \frac{x_i - x_k}{x_k - x_0} (f(x_i) - f(x_i)) = f(x_i).
\]
Similarly, evaluating

\[ q(x) + \frac{x - x_k}{x_k - x_0} (q(x) - p_{k-1}(x)) \]

at \( x_0 \) and \( x_k \) gives \( f(x_0) \) and \( f(x_k) \), respectively.

Thus, both sides of the identity are interpolating the same \( k + 1 \) points and are polynomials with a degree at most \( k \). Thus, their difference is a polynomial of degree at most \( k \) that vanishes at \( k + 1 \) points. Hence, the two sides are equal.

Finally, as the coefficient of \( x^{k-1} \) in \( q \) is \( f[x_1, x_2, \ldots, x_k] \) and that of \( x^{k-1} \) in \( p_{k-1} \) is \( f[x_0, x_1, \ldots, x_{k-1}] \), we have that the coefficient of \( x^k \) on the right hand side of the equation is

\[ \frac{f[x_1, x_2, \ldots, x_k] - f[x_0, x_1, \ldots, x_{k-1}]}{x_k - x_0}. \]
The invariance property of the divided differences

Notice that $f[x_0, x_1, \ldots, x_k]$ is not changed if the nodes $x_0, x_1, \ldots, x_k$ are permuted. Thus, e.g., we have

$$f[x_0, x_1, x_2] = f[x_1, x_2, x_0].$$

The reason is that $f[x_0, x_1, x_2]$ is the coefficient of $x^2$ in the quadratic polynomial $p_2$ interpolating $f$ at $x_0, x_1$ and $x_2$, whereas $f[x_1, x_2, x_0]$ is the coefficient of $x^2$ in the quadratic polynomial $q_2$ interpolating $f$ at $x_1, x_2$ and $x_0$. Due to the uniqueness and existence theorem of polynomial interpolation these two polynomials are of course the same.

This implies the following theorem:

**Theorem (Recursive Property of the Divided Differences)**

The divided difference $f[x_0, x_1, \ldots, x_k]$ is invariant under all permutations of the arguments $x_0, x_1, \ldots, x_k$. 
As the variables \( x_0, x_1, \ldots, x_k \) and \( k \) are arbitrary, the recursive formula can also be written as

\[
f[x_i, x_{i+1}, \ldots, x_{j-1}, x_j] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_j] - f[x_i, x_{i+1}, \ldots, x_{j-1}]}{x_j - x_i}
\]

The first three divided differences are thus

\[
f[x_i] = f(x_i) \\
f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i} \\
f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}.
\]
As an illustration for $n = 3$, this leads to the following divided difference table for an arbitrary function $f$:

\[
\begin{array}{c|cccc}
  x & f[] & f[ , ] & f[ , , ] & f[ , , , ] \\
  \hline
  x_0 & f[x_0] & \rightarrow & f[x_0, x_1] & \rightarrow & f[x_0, x_1, x_2] & \rightarrow & f[x_0, x_1, x_2, x_3] \\
  x_1 & f[x_1] & \rightarrow & f[x_1, x_2] & \rightarrow & f[x_1, x_2, x_3] & \rightarrow & f[x_1, x_2, x_3] \\
  x_2 & f[x_2] & \rightarrow & f[x_2, x_3] & \rightarrow & f[x_2, x_3] & \rightarrow & f[x_2, x_3] \\
  x_3 & f[x_2] & & & & & & \\
\end{array}
\]

In this table, the coefficients along the top diagonal are the ones needed to form the Newton form of the interpolating polynomial $p_3$. 
Example: Set-up of a divided difference table

Example

Construct a divided difference table for the function $f$ given in the following data table, and write out the Newton form of the interpolating polynomial:

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>$\frac{3}{2}$</th>
<th>0</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>3</td>
<td>$\frac{13}{4}$</td>
<td>3</td>
<td>$\frac{5}{3}$</td>
</tr>
</tbody>
</table>

We apply the just derived formulas for the divided differences:

$$f[x_i] = f(x_i)$$
$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$
$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}, \text{ and so on}$$
Example: Set-up of a divided difference table

Example [cont.]

Thus, for

\[
\begin{array}{c|cccc}
  x & 1 & \frac{3}{2} & 0 & 2 \\
  \hline
  f(x) & 3 & \frac{13}{4} & 3 & \frac{5}{3}
\end{array}
\]

the first entry of the divided difference table for \( f \) reads as

\[
f[1, \frac{3}{2}] = \frac{f[\frac{3}{2}] - f[1]}{\frac{3}{2} - 1} = \frac{\frac{13}{4} - 3}{\frac{3}{2} - 1} = \frac{1}{2},
\]

and, for instance, after completion of the columns for \( f[] \) and \( f[ , ] \) the first entry \( f[ , , ] \) is given as

\[
f[1, \frac{3}{2}, 0] = \frac{f[\frac{3}{2}, 0] - f[1, \frac{3}{2}]}{0 - 1} = \frac{\frac{1}{6} - \frac{1}{2}}{0 - 1} = \frac{1}{3}.
\]
Example [cont.]

Finally, the complete divided difference table for $f$ reads as

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f[\ ]$</th>
<th>$f[\ , \ ]$</th>
<th>$f[\ , \ , \ ]$</th>
<th>$f[\ , \ , \ , \ ]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3/2</td>
<td>$\frac{13}{4}$</td>
<td>$\frac{1}{2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{3}$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$\frac{5}{3}$</td>
<td>$\frac{2}{3}$</td>
<td>$\frac{5}{3}$</td>
<td></td>
</tr>
</tbody>
</table>

Thus, we obtain

$$p_3(x) = 3 + \frac{1}{2}(x - 1) + \frac{1}{3}(x - 1)(x - \frac{3}{2}) - 2(x - 1)(x - \frac{3}{2})x.$$
Classroom Problem

Construct a divided difference table for the function $f$ given in the following data table, and write out the Newton form of the interpolating polynomial:

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>−4</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>3</td>
<td>13</td>
<td>−23</td>
</tr>
</tbody>
</table>

Use the just derived formulas for the divided differences:

$$
\begin{align*}
  f[x_i] &= f(x_i) \\
  f[x_i, x_{i+1}] &= \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i} \\
  f[x_i, x_{i+1}, x_{i+2}] &= \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}.
\end{align*}
$$
## Classroom Problem [Solution]

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f[]$</th>
<th>$f[,]$</th>
<th>$f[,]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-4</td>
<td>13</td>
<td></td>
<td>-2</td>
</tr>
<tr>
<td>0</td>
<td>-23</td>
<td></td>
<td>7</td>
</tr>
</tbody>
</table>

Thus we obtain

$$p_2(x) = 3 - 2(x - 1) + 7(x - 1)(x + 4).$$
Interpolation with Monomials and the Vandermode Matrix
Another view of interpolation is that for a given set of \( n + 1 \) data points

\[
\begin{array}{c|c|c|c|c|c}
  x & x_0 & x_1 & x_2 & \ldots & x_n \\
  f(x) = y & y_0 & y_1 & y_2 & \ldots & y_n \\
\end{array}
\]

we want to express an interpolating function \( f(x) \) as a linear combination of a set of basis functions \( \varphi_0, \varphi_1, \ldots, \varphi_n \) such that

\[
f(x) \approx c_0 \varphi_0(x) + c_1 \varphi_1(x) + \cdots + c_n \varphi_n(x),
\]

where the coefficients \( c_0, c_1, \ldots, c_n \) are to be determined.

We want the function \( f \) to interpolate the data \( (x_i, y_i) \), i.e., we want to have linear equations in \( c_0, c_1, \ldots, c_n \) of the form

\[
f(x_i) = c_0 \varphi_0(x_i) + c_1 \varphi_1(x_i) + \cdots + c_n \varphi_n(x_i) = y_i,
\]

for each \( i = 0, 1, \ldots, n \).
This is a system of linear equations

$$Ac = y,$$

where the entries of the coefficient matrix $A$ are given by $a_{i,j} = \varphi_j(x_i)$, which is the value of the $j$th basis function evaluated at the $i$th data point. The right-hand side vector $y$ contains the known data values $y_i$, and the components of the vector $c$ are the unknown coefficients $c_i$.

Thus, in principle interpolation is nothing else than solving a linear system . . .
The natural basis for the vector space $\mathcal{P}_n$ of all polynomials up to degree $n$ consists of **monomials**

$$
\varphi_0(x) = 1, \quad \varphi_1(x) = x, \quad \varphi_2(x) = x^2, \quad \ldots \quad \varphi_n(x) = x^n.
$$

Consequently, a interpolation polynomial $p_n$ has the form

$$
p_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n,
$$

and the associates linear system $A c = y$ reads as

$$
\begin{pmatrix}
1 & x_0 & x_0^2 & \cdots & x_0^n \\
1 & x_1 & x_1^2 & \cdots & x_1^n \\
1 & x_2 & x_2^2 & \cdots & x_2^n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_n & x_n^2 & \cdots & x_n^n \\
\end{pmatrix}
\begin{pmatrix}
c_0 \\
c_1 \\
c_2 \\
\vdots \\
c_n \\
\end{pmatrix}
=
\begin{pmatrix}
y_0 \\
y_1 \\
y_2 \\
\vdots \\
y_n \\
\end{pmatrix}.
$$
Sketch of the first few monomials on \([-1, 1]\):

\begin{align*}
\phi_0 &= 1 \\
\phi_1 &= x \\
\phi_2 &= x^2 \\
\phi_3 &= x^3 \\
\phi_4 &= x^4
\end{align*}
The coefficient matrix is the well-known **Vandermonde matrix**. We already know that it is regular provided the points \( x_0, x_1, \ldots, x_n \) are distinct. So, in principle, we can solve the polynomial interpolation problem.

Although, the Vandermonde matrix is invertible, we know that it is ill-conditioned as \( n \) increases: For large \( n \), the monomials are less distinguishable from one another (see the figure on the previous slide). Thus, the columns of the Vandermonde matrix become nearly dependent in this case. Moreover, high-degree polynomials often oscillate widely and are highly sensitive to small changes in the data.

Up to now, we have discussed three choices for the basis functions of \( \mathcal{P}_n \): the Lagrange polynomials \( l_i(x) \), the Newton polynomials \( \pi_i \), and the monomials. It turns out that there are better choices for the basis functions; namely, the Chebysev polynomials have more desirable features.
The Chebysev polynomials play an important role in mathematics because they have several special properties such as the recursive relation

\[ T_0(x) = 1, \quad T_1(x) = x \]
\[ T_i(x) = 2xT_{i-1}(x) - T_{i-2}(x) \]

for \( i = 2, 3, 4, \ldots \). Thus, the first six Chebysev polynomials are

\[ T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \]
\[ T_4(x) = 8x^4 - 8x^2 + 1, \quad T_5(x) = 16x^5 - 20x^3 + 5x. \]

The Chebysev polynomials are usually employed on the interval \([-1, 1]\). With change of variable, they can be used on any interval. E.g., substituting \( x \) by \( \cos(t) \) allows for their use on the interval \([0, 2\pi]\).
Sketch of the first few Chebysev polynomials on $[-1, 1]$: 

- $T_0(x)$
- $T_1(x)$
- $T_2(x)$
- $T_3(x)$
- $T_4(x)$
- $T_5(x)$
One of the important properties of the Chebysev polynomials is the **equal oscillation property**. Notice in the previous figure that successive extreme points of the Chebysev polynomials are equal in magnitude and alternate in sign.

This property tends to distribute the error uniformly when Chebysev polynomials are used as basis functions.

In polynomial interpolation for continuous functions, it is particularly advantageous to select as the interpolation points the roots or the extreme points of a Chebysev polynomial. This causes the maximum error over the interval of the interpolation to be minimized. (We will illustrate this effect next time.)
Summary & Outlook
The **Newton form** of the interpolation polynomial is

\[
p_n(x) = \sum_{i=0}^{n} a_i \prod_{j=0}^{i-1} (x - x_j).
\]

with **divided differences**

\[
a_i = f[x_0, x_1, \ldots, x_i] = \frac{f[x_1, x_2, \ldots, x_i] - f[x_0, x_1, \ldots, x_{i-1}]}{x_i - x_0}
\]

The Lagrange and the Newton form are just two different representations of the unique polynomial \( p \) of degree \( n \) that interpolates a table of \( n + 1 \) pairs of points \( (x_i, f(x_i)) \) for \( i = 0, 1, \ldots, n \).
We illustrate the Lagrange and Newton form with the aid of a small table for $n = 2$:

\[
\begin{array}{c|c|c|c}
  x & x_0 & x_1 & x_2 \\
  \hline
  f(x) & f(x_0) & f(x_1) & f(x_2)
\end{array}
\]

The Lagrange interpolation polynomial is

\[
p_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)
\]

Clearly, $p_2(x_0) = f(x_0)$, $p_2(x_1) = f(x_1)$ and $p_2(x_2) = f(x_2)$.  


Next, we form the divided difference table

<table>
<thead>
<tr>
<th>x</th>
<th>f[]</th>
<th>f[,]</th>
<th>f[,]</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>x₀</td>
<td>f[x₀]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x₁</td>
<td>f[x₁]</td>
<td></td>
<td></td>
<td>f[x₀, x₁]</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x₂</td>
<td>f[x₂]</td>
<td></td>
<td></td>
<td>f[x₁, x₂]</td>
</tr>
</tbody>
</table>

Using the divided difference entries from the top diagonal, we have

\[ p_2(x) = f[x₀] + f[x₀, x₁](x - x₀) + f[x₀, x₁, x₂](x - x₀)(x - x₁). \]

Again, it can be easily shown that \( p_2(x₀) = f(x₀), p_2(x₁) = f(x₁) \) and \( p_2(x₂) = f(x₂). \)
Please, prepare these short exercises for the next lecture:

1. **Page 175, exercise 10 a**
   Use a divided differences table to construct Newton’s interpolation polynomial for the following data:

   \[
   \begin{array}{c|cccc}
   x & 0 & 2 & 3 & 4 \\
   f(x) & 7 & 11 & 28 & 63 \\
   \end{array}
   \]

2. **Page 175, exercise 10 b**
   Without simplifying the Newton interpolation polynomial obtained above, write this polynomial in nested form for easy evaluation.
Please, prepare these short exercises for the next lecture:

3. **Page 176, exercise 22**
   Find a polynomial of least degree that takes these values:

   \[ f(x) = y \]

   \[
   \begin{array}{c|c|c|c|c|c}
   x & 1.73 & 1.82 & 2.61 & 5.22 & 8.26 \\
   f(x) = y & 0 & 0 & 7.8 & 0 & 0 \\
   \end{array}
   \]

   *Hint:* Rearrange the table so that the non-zero value of \( f(x) \) is the last entry, or think of some better way.