CPSC 314
Assignment 3
Solution

Due Monday Nov 2nd, 2015, in class

Answer the questions in the spaces provided on the question sheets. If you run out of space for an answer, use separate pages and staple them to your assignment.

Name: ______________________________

Student Number: ______________________________

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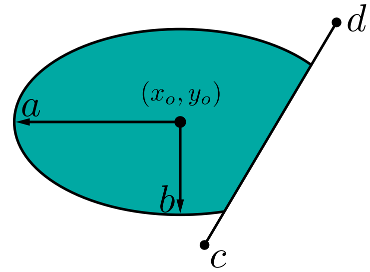
1. (5 points) Scan Conversion
   Give the pseudocode for scan converting the region shown to the right.

   It is the portion of the ellipse on the left side of the line defined by fixed points $c = (1, 0)$ and $d = (2, 2)$. The ellipse is centred at (x_o, y_o) with a major axis of length $2a$ and a minor axis of length $2b$. Use implicit equations to develop your solution.

   Solution:

   Line equations are in the form $Ax + By + C = 0$. To calculate line cd's equation, we just substitute the x and y in the equation with c's and d's coordinates, then we get $-2Bx + By + 2B = 0$, where $B \neq 0$.

   Let $E_{cd}(x, y) = 2x - y - 2$, then for any point i with coordinate (x_i, y_i), if $E_{cd}(x_i, y_i) < 0$, i is on the left side of cd; if $E_{cd}(x_i, y_i) = 0$, i is on the boundary; if $E_{cd}(x_i, y_i) > 0$, i is on the right side of cd.

   For ellipse, let $E_{ellipse}(x_i, y_i)$ be its equation, we have $E_{ellipse}(x_i, y_i) = (\frac{x_i - x_0}{a})^2 + (\frac{y_i - y_0}{b})^2 - 1$. If $E_{ellipse}(x_i, y_i) < 0$, point i is inside this ellipse; if $E_{ellipse}(x_i, y_i) = 0$, point i is on the boundary; if $E_{ellipse}(x_i, y_i) > 0$, point i is outside.

   So if the point i on the screen satisfied both $E_{ellipse}(x_i, y_i) < 0$ and $E_{cd}(x, y) < 0$, we will paint that pixel. But testing for pixels on the entire screen costs too much. Actually, we only need to test pixels in the bounding box of the region.

   To compute a bounding box, we just need to calculate the largest and smallest x and y value of the pixels in the region. In this case, except for the right bound, others are quite obvious: $y_{top} = y_0 - b$, $y_{bottom} = y_0 + b$, $x_{left} = x_0 - a$. (Note that the origin of the screen is at upleft.) For the right bound, we need to know the upright intersection points between the line and the ellipse. So we combine their equations and solve, and we will get the right intersection point's x coordinate x_{right}.

   Here's the pseudocode of the algorithm:

   for x_i *from* $x_0 - a$ *to* x_{right} **do**
       for y_i *from* $y_0 - b$ *to* $y_0 + b$ **do**
           if $E_{ellipse}(x_i, y_i) < 0 \&\& E_{cd}(x_i, y_i) < 0$ **then**
               paint pixel(x_i, y_i)
           end
       end
   end

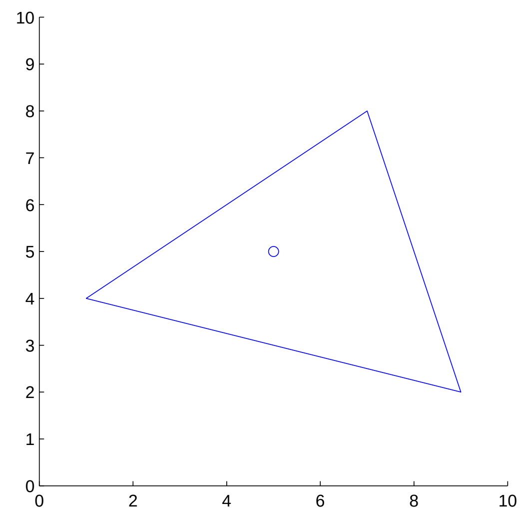
2. (9 points) Interpolation

A triangle has device coordinates $P_1 = (1, 4)$, $P_2 = (9, 2)$, and $P_3 = (7, 8)$. You wish to interpolate a value v for point $P = (5, 5)$, given the value of v at the vertices:

$v_1 = 2, v_2 = 8, v_3 = 5$.

(a) (1 point) Sketch the triangle and the point P.

Solution:



(b) (4 points) Develop a plane equation for v as a function of the x and y, i.e. $Ax + By + Cv + D = 0$. To do this, determine the constraints based on the given triangle points and their respective values v_i. You can use Matlab or an online linear solver to solve a set of linear equations for your plane parameters. Compute v for point P using the plane equation.

Solution:

In order to express v by x and y, we need to solve the plane equation $Ax + By + Cv + D = 0$. Substituting the x, y and v in the equation by P_1, P_2 and P_3's values, we get a linear system:

$$\begin{pmatrix} 1 & 4 & 1 \\ 9 & 2 & 1 \\ 7 & 8 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ D \end{pmatrix} = \begin{pmatrix} -2C \\ -8C \\ -5C \end{pmatrix}.$$

Solving this linear system gives us $A = -0.6818C, B = 0.2727C, D = -2.4091C$. Here C couldn't be zero, because in that case, we no longer have a plane equation. So we can let $C = 1$, then we have:

$$v = 0.6818x - 0.2727y + 2.4091$$

Then we substitute the x and y by P's coordinate, and it gives us $v_P = 4.4546$

(c) (4 points) Compute the barycentric coordinates for point P. Compute v for point P using the Barycentric coordinates.

Solution:

According to the definition of barycentric coordinate, we can express P's coordinate using the coordinates of P_1, P_2, and P_3 with:

$$P = \frac{A_{PP_2P_3}}{A_{P_1P_2P_3}} P_1 + \frac{A_{PP_1P_3}}{A_{P_1P_2P_3}} P_2 + \frac{A_{PP_1P_2}}{A_{P_1P_2P_3}} P_3$$

where $A_{XYZ} = \frac{1}{2}||\vec{XY} \times \vec{XZ}||$. So we have:

$$\begin{aligned}
&A_{P_1P_2P_3} = 0.5||(8,-2,0) \times (6,4,0)|| = 22 \\
&A_{PP_1P_2} = 0.5||(-4,-1,0) \times (4,-3,0)|| = 8 \\
&A_{PP_2P_3} = 0.5||(4,-3,0) \times (2,3,0)|| = 9 \\
&A_{PP_1P_3} = 0.5||(-4,-1,0) \times (2,3,0)|| = 5 \\
&P = \frac{9}{22} P_1 + \frac{5}{22} P_2 + \frac{8}{22} P_3
\end{aligned}$$

which means the barycentric coordinate of P is $(\frac{9}{22}, \frac{5}{22}, \frac{8}{22})$, and we can now calculate:

$$v_P = \frac{9}{22} v_1 + \frac{5}{22} v_2 + \frac{8}{22} v_3 = 4.4546$$

the same results with (b).

3. (10 points) Clipping
   Suppose that a perspective view-volume is defined by

   `near = -1, far = -5, bottom = -5, top = 5, left = -8, right = 8.`

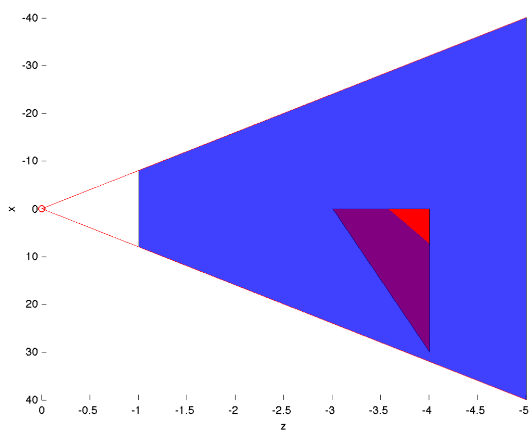
   Consider the triangle defined by the view coordinate frame coordinates

$$P_1 = (0, 2, -3), \qquad P_2 = (30, -10, -4), \qquad P_3 = (0, 30, -4).$$

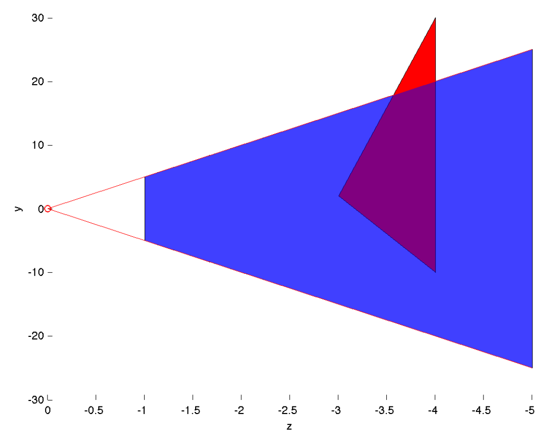
   (a) (3 points) Sketch a side-view and top-view of the view-volume and the triangle.

   Solution:

   Topview:



   Sideview:



(b) (3 points) Determine if view-frustum culling can be applied to the triangle, i.e., if any vertices are "outside" with respect to any one of the six view frustum planes. Use the implicit plane equation

$$Ax + By + Cz + D = 0$$

Hint: remember what we discussed in class about plane equation and its sign.

Hint 2: If you find you have to do a lot of tedious calculations, think more.

Solution:

So we need to calculate the plane equations for the 6 surfaces of the frustum to test their intersection with the triangle.

For the top plane, we already know three points on it. They are $(0, 0, 0)$, $(8, 5, -1)$, and $(-8, 5, -1)$. Because the top plane is not parallel to the y axis, we can let $B = -1$ and the plane equation becomes $Ax + Cz + D = y$. Then we can construct the following linear system and solve by plugging in the coordinates of the three points into the function:

$$\begin{pmatrix} 0 & 0 & 1 \\ 8 & -1 & 1 \\ -8 & -1 & 1 \end{pmatrix} \begin{pmatrix} A \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix}.$$

Then we got the top plane's equation $y = -5z$.

Similarly, we can calculate out the bottom plane's equation $y = 5z$, the left plane's equation $x - 8z = 0$, the right plane's equation $x + 8z = 0$, the near plane's equation $z = -1$, and the far plane's equation $z = -5$.

Then we can form a constraint for points inside the frustum:

$$\begin{cases} 5z \leq y \leq -5z \\ 8z \leq x \leq -8z \\ -5 \leq z \leq -1 \end{cases}$$

So there's only $P_3 = (0, 30, -4)$ not in the frustum, because it fails the test $5z \leq y \leq -5z$, which means P_3 is just above the frustum. This also tells us the polygon inside the frustum is a quadrangle, which is consisted of P_1, P_2 and the two intersection points on the top plane.

(c) (1 point) Based on your work for the question above, determine the view-frustum planes that the triangle intersects.

Solution:

See solution in 3(b).

(d) (3 points) Compute the final clipped polygon of the triangle in VCS. Show your work.

Solution:

To calculate the two intersection points, we need P_3P_1's and P_3P_2's line equations. They are $(0, 30, -4) + t(0, -28, 1)$ where $0 \leq t \leq 1$ for P_3P_1 and $(0, 30, -4) + t(30, -40, 0)$ where $0 \leq t \leq 1$ for P_3P_2.[1] Combining them with the top plane's equation $y = -5z$ gives us the two intersection points $P_4 = (0, \frac{410}{23}, -\frac{82}{23})$ and $P_5 = (\frac{15}{2}, 20, -4)$. So the clipped polygon is a quadrangle consisted of $P_1P_2P_4P_5$.

[1] See http://tutorial.math.lamar.edu/Classes/CalcIII/EqnsOfLines.aspx for 3D line equation.

4. (6 points) Local Illumination

Sketch the illumination for the following scene when computed using the Phong illumination model. The scene is viewed from above using an orthographic projection and is lit by the single light source L. Draw 4 curves (overlay these in the graph below), one for each of ambient, diffuse, specular, and total illumination. The Phong illumination model is given by:

$$I = k_a I_a + k_d I_d (N \cdot L) + k_s I_s (R \cdot V)^n$$

with the following values:

$$I_a = I_d = I_s = 1.0, k_a = 0.3, k_d = 0.7, k_s = 0.9, n = 50$$

Hint: it may be useful to draw vertical lines marking critical points the surface. We do not expect numerically accurate plots, but do expose the characteristics of the curves.

Solution:

For ambient, the illumination curve should be a horizontal line $y = C_a = k_a * I_a = 0.3$

For diffuse, we need to divide the shape into 4 parts, S_1 on the left of the semicircle, S_2 for the left part of the semicircle that can be lit, S_3 for the rightmost part of the semicircle that can't be lit, S_4 on the right of the semicircle.

For S_1, from left to right, the angle between L and N decreases from around $70°$ to $60°$, so C_d should increase from $0.7cos(70°)$ to $0.7cos(60°)$, approximately 0.24 to 0.35.

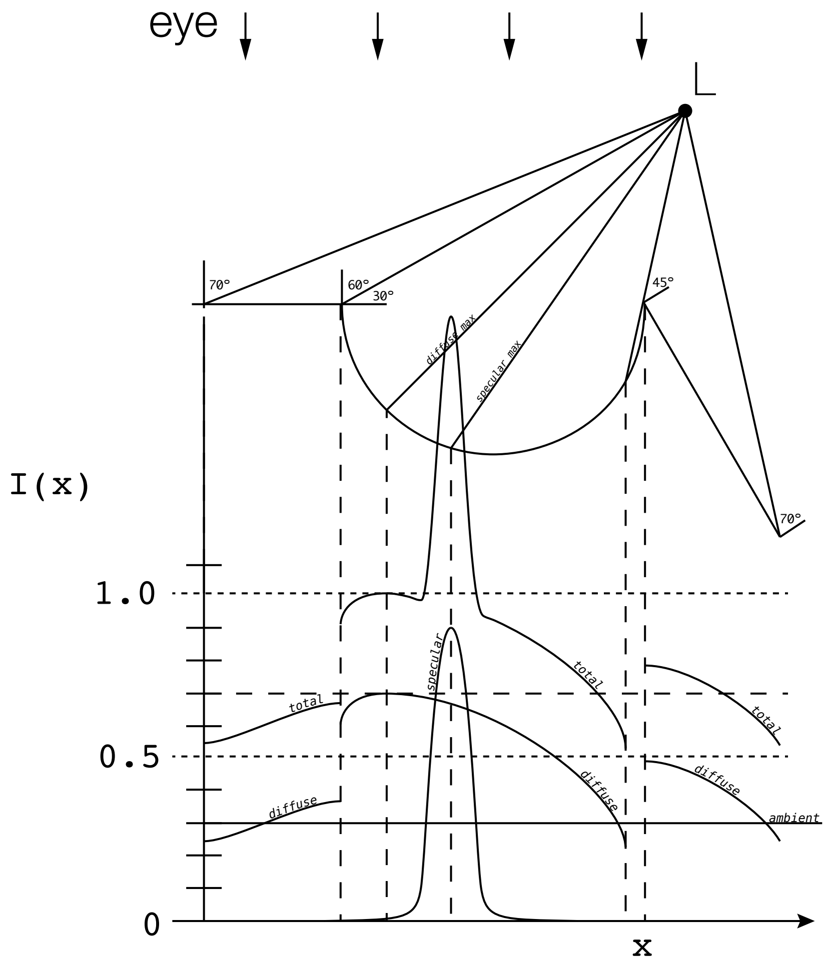
For S_2, the angle between L and N first decreases from around $30°$ to $0°$ and then increases from $0°$ to $70°$, so C_d should first increase from $0.7cos(30°)$ to $0.7cos(0°)$ and then decrease to $0.7cos(70°)$, approximately from 0.61 to 0.7 then to 0.24.

For S_3, $C_d = 0$ since it is inside the shadow.

For S_4, from left to right, the angle between L and N increases from around $45°$ to $70°$, so C_d should decrease from $0.7cos(45°)$ to $0.7cos(70°)$, approximately 0.49 to 0.24.

For specular, the peak value is at where V and $L_{reflect}$ has angle $0°$, which can be calculated as $C_s = 0.9(1.0)^{50} = 0.9$. Then C_s will rapidly decrease in both direction.

For the total illumination, we just add up the ambient, diffuse, and specular illumination curves.



For those of you interested in what the specular falloff curve with $n = 50$ would look like, here you go: